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Dynamic equations of motion for a rigid or deformable body in an arbitrary non-uniform potential flow field

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In this paper we present a general method for calculating the hydrodynamic loads (forces and moments) acting on a deformable body moving with six degrees of freedom in a *non-uniform* ambient potential flow field. The corresponding expressions for the force and moment are given in a moving (body-fixed) coordinate system. The newly derived system of nonlinear differential equations of motion is shown to possess an important antisymmetry property. As a consequence of this special property, it is demonstrated that the motion of a rigid body embedded into a stationary flow field always renders a first integral. In a similar manner, we show that the motion of a deformable body in the presence of an arbitrary ambient flow field is Hamiltonian. A few practical applications of the proposed formulation for quadratic shapes and for weakly non-uniform external fields are presented. Also discussed is the self-propulsion mechanism of a deformable body moving in a non-uniform stationary flow field. It leads to a new parametric resonance phenomenon.

1. Introduction

One of the classical problems in fluid mechanics is that of evaluating the hydrodynamic reactions on a rigid body moving unsteadily in an unbounded perfect fluid which is otherwise at rest. The elegant Kirchhoff–Lagrange formulation expresses the forces and moments acting on such a body in terms of its six velocities and the added-mass tensor (e.g. Lamb 1945, Ch. 6; Kochin, Kibel & Rose 1964, Ch. 7; Milne-Thomson 1968, Ch. 17). These equations generalize the Euler equations of motion of a rigid body in vacuum. The resulting dynamic system consists of six ordinary differential equations, written in a coordinate system moving with the body. One of the remarkable properties of the Kirchhoff equations is that they account for the influence of the fluid (an infinite-dimensional system) only through an additional tensorial parameter (the added-mass tensor), depending on the body's geometry. The same method can also be directly extended to the case of a body embedded in a *uniform* flow, by virtue of the Galilean invariance principle.

However, the problem of generalizing the Kirchhoff formalism for a body moving in an arbitrary *non-uniform* ambient flow field is far from being trivial and has not yet been resolved. So far, only the case of a *weak* flow non-uniformity (i.e. when the characteristic length scale of the flow non-homogeneity is much larger than the characteristic length scale of the body) has been considered (first by Taylor 1928) for a *stationary* rigid body placed in a *steady* ambient flow. Recently Galper & Miloh (1994) provided an extension of the Kirchhoff–Lagrange method for *deformable* shapes moving with six degrees of freedom in a time-dependent *weakly non-uniform* imposed flow field.

The purpose of this paper (motivated by recent developments in bubble dynamics and other dispersed systems) is to present a concise analysis of the hydrodynamical problem corresponding to a deformable body moving unsteadily in an *arbitrary* time-dependent ambient perfect flow field. Using the momentum approach, we derive a new dynamical system of equations which govern the motion of a deformable body in an arbitrary non-uniform flow field which can still be reduced to a set of six nonlinear ordinary differential equations of the second order. The effect of flow non-uniformity enters into the formulation through an explicit dependence of the various coefficients on a quadratic vector functional of the external velocity field $\boldsymbol{V}$. There is another term representing the coupling between the velocity of the body and some linear functional of $\boldsymbol{V}$. For the case of a quiescent flow field (i.e. $\boldsymbol{V} = 0$), these two terms vanish and the classical form of the Kirchhoff–Lagrange equation (e.g. Milne-Thomson 1968, eq. 17.43) is recovered. On the other hand, if the flow non-uniformity is assumed to be *weak*, the recently derived expressions of Galper & Miloh (1994) are readily obtained as a limiting case.

It is demonstrated below, that this dynamical system of equations (governing the motion of a deformable body in an arbitrary non-uniform flow), possesses some important antisymmetric property. As a direct consequence of the special symmetry of the dynamical system we prove the existence of a first integral of motion for a *rigid* body moving in a *stationary* non-uniform stream. This first integral is shown to be compatible with an energy conservation principle expressed in Lagrangian coordinates. In turn, it points to the possibility of using Hamiltonian formalism to describe the motion of a deformable body in a non-uniform flow field. The possibility of expressing the Kirchhoff equations (in the absence of an imposed flow) in a canonical Hamiltonian form was first demonstrated by Lamb (1945, Ch. 5). To the best of our knowledge, the exact form of the Hamiltonian for a deformable body moving in an arbitrary *non-uniform* flow field, is presented here for the first time.

The Kelvin impulse and Kelvin-impulse couple of the motion of the surrounding fluid, induced by the body, here play the role of the generalized impulses in the corresponding Hamiltonian formalism. The present Hamiltonian formalism may be found useful in the statistical approach to the theory of bubbly liquids (e.g. Zhang & Prosperetti 1994) and for studying the stability of bubble motion. It should be mentioned that a more restricted Hamiltonian formalism has been previously employed by Benjamin (1987) in analysing the motion of a *deformable* bubble embedded in a *quiescent* fluid. The existence of the first integral also implies that the body's six velocities are all bounded and suggests, among other things, a qualitative physical model for estimating the magnitude of the hydrodynamic 'spreading' of rigid particles suspended in a non-uniform ambient flow field.

The general formulation of the problem is outlined in §2. The dynamical system of ordinary differential equations, which govern the motion of a deformable body, is then obtained in §3 (see also Appendices A and B). Some important antisymmetric properties of the resulting matrices are derived and their physical consequences are discussed. The corresponding equations of motion for the case of a deformable body moving in a *weakly* non-uniform flow field are then obtained in §4 as a limiting case of the general dynamical system. Several other physical examples concerning impulsive bubble dynamics and some new effects of self-propulsion of deformable shapes are presented and analysed in §5. An energy-based approach is further elaborated in §6–§8. Thus, by utilizing the fact that the coefficient matrix in the governing equations of motion in §3 is antisymmetric, we derive in §6 the first integral of motion for the case of a *rigid* body moving in a *stationary* flow field.

The present Hamiltonian framework, is further extended in §7 where we derive explicit expressions for the Hamiltonian and for the corresponding conjugated variables for a deformable body embedded in a time-dependent flow field. Appropriate bounds on the body's velocity, as well as some useful expressions for estimating the rate of 'solid spreading' are obtained in §8. The aforementioned formulation is finally applied in §9 for ellipsoidal and spherical shapes, which are frequently encountered in bubble dynamics. A compact degenerate form for the equations of motion of a sphere immersed in an arbitrary flow field is then found in a format equivalent to that corresponding to the motion of a rigid particle in an effective potential force field. The motion of a rigid sphere in a non-uniform stream with spherical symmetry is proven to be completely integrable. Other general results, pertinent to the dynamics of rigid and deformable shapes in a non-uniform flow field, including the interesting effect of self-propulsion, are also presented.

2. General formulation and kinematic preliminaries

2.1. *Force and moment loadings*

Consider an arbitrary unsteady irrotational flow of an incompressible liquid past a moving deformable body, with an ambient stream $\boldsymbol{V} = \nabla\phi(\boldsymbol{X}, t)$. The rectilinear velocity of the body's centroid is $\boldsymbol{U}(t)$ and the angular velocity of its principle axes is denoted by $\boldsymbol{\Omega}(t)$. Measured in a body-fixed coordinate system, the time-dependent surface of the deformable body is given by the equation $S(\boldsymbol{x}, t) = 0$, where $\boldsymbol{x}$ is a Cartesian vector taken in the body-fixed coordinate system. Thus, the normal component of the surface-deformation velocity $\boldsymbol{V}_d(t)$ is defined correspondingly as

$$\boldsymbol{n} \cdot \boldsymbol{V}_d \equiv -\frac{\dot{S}}{|\nabla S|}, \tag{2.1}$$

where the dot represents differentiation with respect to time. The body is *instantaneously* introduced into an ambient stream $\boldsymbol{V} = \nabla\phi$, such that the body's centroid lies at the point $\boldsymbol{X}$ and there are no field singularities of the imposed flow inside the body.

The total velocity potential φ, induced by the presence of the moving deformable body, can be decomposed into

$$\varphi = \phi + \acute{\phi}, \tag{2.2}$$

where $\acute{\phi}$ represents the additional disturbance potential satisfying a proper decay condition at infinity, i.e.

$$\lim_{|\boldsymbol{x}| \to \infty} \acute{\phi}(\boldsymbol{x}) = 0. \tag{2.3}$$

It is important to note that $\acute{\phi}$ is harmonic *outside* S and ϕ is harmonic *inside* S.

The impermeable boundary conditions on the deformable surface S, are

$$\frac{\partial \varphi}{\partial n} = (\boldsymbol{U} + \boldsymbol{\Omega} \wedge \boldsymbol{x} + \boldsymbol{V}_d) \cdot \boldsymbol{n}, \tag{2.4}$$

where $\boldsymbol{n}$ denotes a unit normal vector to $S(t)$ directed outward into the fluid. We use the symbol $\wedge$ for a vector product.

Let us next introduce the outer Green function $G^{(out)}(\boldsymbol{x}, \boldsymbol{y})$ (depending only on the body's geometry) which represents the solution of the following Poisson equation:

$$\nabla^2 G^{(out)}(\boldsymbol{x},\boldsymbol{y}) = 4\pi\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{2.5}$$

The corresponding boundary conditions supplementing (2.5) are written

$$\frac{\partial}{\partial n}G^{(out)}(\boldsymbol{x},\boldsymbol{y})\bigg|_S = 0, \text{ and } \lim_{|\boldsymbol{x}|\to\infty} G^{(out)}(\boldsymbol{x},\boldsymbol{y}) = 0. \tag{2.6}$$

One can also express $\acute{\phi}$ in the common Kirchhoff manner, as a linear decomposition of unit time-dependent potentials,

$$\acute{\phi} = \boldsymbol{U}\cdot\boldsymbol{\Phi} + \boldsymbol{\Omega}\cdot\boldsymbol{\Psi} + \phi_d + \acute{\phi}_0, \tag{2.7}$$

where $\acute{\phi}_0$ denotes the potential response of a *stationary* body. The decomposition (2.7) is subject, by virtue of (2.4), to the following boundary conditions on $S(t)$:

$$\frac{\partial\boldsymbol{\Phi}}{\partial n} = \boldsymbol{n}\bigg|_S; \quad \frac{\partial\boldsymbol{\Psi}}{\partial n} = \boldsymbol{x}\wedge\boldsymbol{n}\bigg|_S; \quad \frac{\partial\phi_d}{\partial n} = -\frac{\dot{S}}{|\nabla S|}\bigg|_S; \quad \frac{\partial\acute{\phi}_0}{\partial n} = -\boldsymbol{V}\cdot\boldsymbol{n}\bigg|_S, \tag{2.8}$$

and a proper decay at infinity.

It is important to note that both $\acute{\phi}$ and $\acute{\phi}_0$ can be expressed only in terms of the body's geometry and the values of the ambient flow field $\boldsymbol{V}$ taken on the body's surface, i.e.

$$\acute{\phi}_0(\boldsymbol{x}) \equiv -\int_S G^{(out)}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{V}(\boldsymbol{y})\cdot\boldsymbol{n}(\boldsymbol{y})\mathrm{d}S(\boldsymbol{y}). \tag{2.9}$$

We prove further that (2.9) implies that the present dynamical problem comprising an interacting body with a fluid which is basically an infinite-degree system, can be generally reduced to a 12 degree system (i.e. 6 generalized coordinates for the body's position and 6 generalized velocities prescribing the body's motion), where the equation of the body's surface and the flow field $\boldsymbol{V}$ play the role of the ambient time-dependent parameters.

Here and in the sequel we will use bold faced letters to denote vectors. Tensors are designated by bold sans serif. For brevity we will also select the density of the fluid to be unity, i.e. $\rho_f = 1$.

The equation $S(\boldsymbol{x},t) = 0$ of the body's surface is given in a coordinate system attached to and moving with the body centroid, which leads to the following geometrical restrictions on S (see Miloh & Galper 1993):

$$\int_S x_i\frac{\dot{S}}{|\nabla S|}\mathrm{d}S = 0, \qquad \int_S x_i x_j\frac{\dot{S}}{|\nabla S|}\mathrm{d}S = 0, \qquad i \neq j = 1,2,3, \tag{2.10}$$

implying that both the position of the body's centroid and the directions of its principal axes are preserved.

The hydrodynamic force $\boldsymbol{F}$ and moment $\boldsymbol{M}$ acting on a deformable body immersed in the potential fluid $\boldsymbol{V} = \nabla\phi$, can then be expressed in the moving system (e.g. Batchelor 1967) as

$$\boldsymbol{F}(\phi) = \frac{\mathrm{d}}{\mathrm{d}t}\int_S(\phi+\acute{\phi})\boldsymbol{n}\mathrm{d}S + \boldsymbol{F}_{st} \tag{2.11}$$

and

$$\boldsymbol{M}(\phi) = \frac{\mathrm{d}}{\mathrm{d}t}\int_S(\phi+\acute{\phi})\boldsymbol{x}\wedge\boldsymbol{n}\mathrm{d}S + \boldsymbol{U}\wedge\int_S(\phi+\acute{\phi})\boldsymbol{n}\mathrm{d}S + \boldsymbol{M}_{st}, \tag{2.12}$$

where the moment is evaluated about the body's centroid $\boldsymbol{X}$. The corresponding expressions for the 'steady' force and moment are

$$\boldsymbol{F}_{st} = \frac{1}{2}\int_S (\nabla\varphi)^2 \boldsymbol{n}\, \mathrm{d}S - \int_S \nabla\varphi(\nabla\varphi \cdot \boldsymbol{n})\, \mathrm{d}S, \tag{2.13}$$

and

$$\boldsymbol{M}_{st} = \frac{1}{2}\int_S (\nabla\varphi)^2 (\boldsymbol{x}\wedge\boldsymbol{n})\, \mathrm{d}S - \int_S \boldsymbol{x}\wedge \boldsymbol{V}(\nabla\varphi \cdot \boldsymbol{n})\, \mathrm{d}S. \tag{2.14}$$

Using the decomposition (2.2) one can express $\boldsymbol{F}_{st}$ and $\boldsymbol{M}_{st}$ in terms of the cross-products of ϕ and $\acute{\phi}$. Indeed, after the substitution of (2.2) into (2.13) and enforcing the Green formula, one gets

$$\frac{1}{2}\int_S (\nabla\phi)^2 \boldsymbol{n}\, \mathrm{d}S - \int_S \nabla\phi(\nabla\phi \cdot \boldsymbol{n})\, \mathrm{d}S = 0, \quad \frac{1}{2}\int_S (\nabla\acute{\phi})^2 \boldsymbol{n}\, \mathrm{d}S - \int_S \nabla\acute{\phi}(\nabla\acute{\phi} \cdot \boldsymbol{n})\, \mathrm{d}S = 0, \tag{2.15}$$

which lead to

$$\boldsymbol{F}_{st} = \int_S (\boldsymbol{V}\cdot\acute{\boldsymbol{V}})\boldsymbol{n}\, \mathrm{d}S - \int_S \left(\boldsymbol{V}(\acute{\boldsymbol{V}}\cdot\boldsymbol{n}) + \acute{\boldsymbol{V}}(\boldsymbol{V}\cdot\boldsymbol{n})\right) \mathrm{d}S, \tag{2.16}$$

where $\acute{\boldsymbol{V}} \equiv \nabla\acute{\phi}$. Using the Gauss theorem within the volume v of the body and interpreting $\Delta\acute{\phi}$ as a sum of δ-functions and their derivatives, one gets

$$\int_S (\boldsymbol{V}\cdot\acute{\boldsymbol{V}})\boldsymbol{n}\, \mathrm{d}S = \int_v (\nabla\cdot(\acute{\phi}\nabla\boldsymbol{V}) + \boldsymbol{V}\cdot\nabla\acute{\boldsymbol{V}})\, \mathrm{d}v. \tag{2.17}$$

Substituting (2.17) into (2.16) leads then to the desired expression

$$\boldsymbol{F}_{st} = \int_S \left(\acute{\phi}\frac{\partial \boldsymbol{V}}{\partial n} - \frac{\partial\acute{\phi}}{\partial n}\boldsymbol{V}\right) \mathrm{d}S. \tag{2.18}$$

In a similar manner, the steady moment can be written as

$$\boldsymbol{M}_{st} = \int_S \left(\acute{\phi}\frac{\partial}{\partial n}(\boldsymbol{x}\wedge\boldsymbol{V}) - \frac{\partial\acute{\phi}}{\partial n}(\boldsymbol{x}\wedge\boldsymbol{V})\right) \mathrm{d}S. \tag{2.19}$$

Making use of the Green theorem within v for (2.18) and (2.19) yields the classical Lagally expressions for the 'steady' force and moment (see Galper & Miloh 1994).

2.2. *Moving coordinate system*

The dynamical variables $\boldsymbol{U}$, $\boldsymbol{\Omega}$ and $\boldsymbol{V}$ are all measured in the *laboratory* (inertial) coordinate system and when substituted into (2.11) and (2.12), must be transferred to the *moving* (body-fixed) coordinate system. Correspondingly, in order to distinguish between the two systems, an asterisk will be used in what follows to denote a tensor referred to the *laboratory* coordinate system.

We introduce an orthogonal operator $\mathbf{Q}(t)$ (a 3×3 matrix), which instantaneously connects the body-fixed and the laboratory coordinate systems by the following transformation:

$$\mathbf{Q}\mathbf{Q}^T = \hat{\mathbf{1}}, \quad \boldsymbol{a}^* = \mathbf{Q}(t)\boldsymbol{a}, \quad \boldsymbol{a} = \mathbf{Q}^T(t)\boldsymbol{a}^*. \tag{2.20}$$

Here $\hat{\mathbf{1}}$ represents a unit matrix, the superscript T represents transpose operation, $\boldsymbol{a}^*$ denotes an arbitrary vector measured in the laboratory system and $\boldsymbol{a}$ is the same vector referred to the body-fixed coordinate system. In a similar way one also has

$$\boldsymbol{V}^* = \mathbf{Q}\boldsymbol{V}, \quad \mathbf{E}^* \equiv \nabla_{X^*}\boldsymbol{V}^*(\boldsymbol{X}^*) = \mathbf{Q}\mathbf{E}\mathbf{Q}^T, \tag{2.21}$$

where $\boldsymbol{E}^*$ and $\boldsymbol{E}$ are the rate-of-strain tensors of the ambient flow evaluated in the laboratory and moving coordinate system respectively. It is also recalled that $\boldsymbol{Q}^* = \boldsymbol{Q}$.

At this stage we choose to use isomorphism τ between 3-vectors and 3×3 antisymmetric matrices (e.g. Marsden 1992), defined as

$$\tau(\boldsymbol{a}) = \boldsymbol{A}; \quad A_{ik} \equiv -\epsilon_{ikl}a_l, \quad \text{with the inverse} \quad \tau^{-1}(\boldsymbol{A}) = \boldsymbol{a}; \quad a_l \equiv -\tfrac{1}{2}\epsilon_{ikl}A_{ik}, \tag{2.22}$$

where ϵ_{ijk} is the permutation tensor. It can be thus shown that

$$\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t} = \boldsymbol{Q}\hat{\boldsymbol{\Omega}}; \quad \text{and} \quad \frac{\mathrm{d}\boldsymbol{Q}^T}{\mathrm{d}t} = -\hat{\boldsymbol{\Omega}}\boldsymbol{Q}^T, \tag{2.23}$$

where the 3×3 antisymmetric tensor $\hat{\boldsymbol{\Omega}}$ is defined below as

$$\hat{\boldsymbol{\Omega}} \equiv \tau(\boldsymbol{\Omega}), \quad \text{and hence} \quad \hat{\boldsymbol{\Omega}}\boldsymbol{a} \equiv \boldsymbol{\Omega} \wedge \boldsymbol{a}. \tag{2.24}$$

Note, that under the constraints (2.10), one can consider the variables $\boldsymbol{X} = \boldsymbol{Q}\boldsymbol{X}^*$ and $\boldsymbol{Q}$ (see Aref & Jones 1993) combined with the prescribed surface equation $S(\boldsymbol{x}, t) = 0$ as the generalized coordinates of the deformable body. The orthogonality of $\boldsymbol{Q}$ must be treated in this case as a weak constraint imposed on the system (see Dirac 1964).

The initially prescribed flow field $\boldsymbol{V}^*$ is determined at points $\boldsymbol{X}^* + \boldsymbol{x}^*$ of the laboratory coordinate system. To evaluate the same vector $\boldsymbol{V}$ at points $\boldsymbol{x}$ in the attached system, one has to first construct the corresponding point $\boldsymbol{Q}\boldsymbol{x}$ in the laboratory system, find the vector $\boldsymbol{V}^*(\boldsymbol{Q}\boldsymbol{x})$ and then map it, using the operator $\boldsymbol{Q}^T$, back into the body-attached (moving) system, namely

$$\boldsymbol{V}(\boldsymbol{x}, t) \equiv \boldsymbol{Q}^T \boldsymbol{V}^* \left(\boldsymbol{X}^*(t) + \boldsymbol{Q}\boldsymbol{x}, t\right). \tag{2.25}$$

Finally, taking the absolute time derivative of (2.25) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{V}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{V}}{\partial t} - \boldsymbol{\Omega} \wedge \boldsymbol{V} + \boldsymbol{E}\boldsymbol{U} + \boldsymbol{E}(\boldsymbol{\Omega} \wedge \boldsymbol{x}), \tag{2.26}$$

which, for $\boldsymbol{E} = 0$ (i.e. for a uniform external flow), reduces to the well-known relationship between accelerations measured in inertial and rotating coordinate systems.

3. The dynamical system of equations of motion

3.1. *The classical Kirchhoff equations for a deformable body*

The classical Kirchhoff equations govern the motion of a rigid body in a *quiescent* flow (e.g. Milne-Thomson 1968, Ch. 17). In this section we present a generalization of the classical Kirchhoff equations for the case of a *deformable* body moving in an *arbitrary* imposed perfect flow field.

Consider a deformable body embedded in a surrounding fluid which is otherwise at rest. The corresponding Kirchhoff force and moment equations can then be written (see Appendix A) as,

$$\frac{\mathrm{d}}{\mathrm{d}t}(v\rho_b \boldsymbol{U}) + \boldsymbol{F}^{(q)} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{I}\boldsymbol{\Omega}) + \boldsymbol{M}^{(q)} = 0. \tag{3.1}$$

Here $-\boldsymbol{F}^{(q)}$ and $-\boldsymbol{M}^{(q)}$ denote the external force and moment exerted on the body expressed in the attached coordinate system, $\boldsymbol{I}$ is the body's inertia tensor, ρ_b is the specific density of the homogeneous body and v is the volume of the body. The force $\boldsymbol{F}^{(q)}$ and the moment $\boldsymbol{M}^{(q)}$ can be naturally split into a force $\boldsymbol{F}^{(q)}_{(r)}$ and a moment $\boldsymbol{M}^{(q)}_{(r)}$, which arise from the rigid body motion, and a force $\boldsymbol{F}^{(q)}_{(d)}$ and a moment $\boldsymbol{M}^{(q)}_{(d)}$ which result from the body's pure deformation, i.e.

$$\boldsymbol{F}^{(q)} = \boldsymbol{F}^{(q)}_{(r)} + \boldsymbol{F}^{(q)}_{(d)}, \qquad \boldsymbol{M}^{(q)} = \boldsymbol{M}^{(q)}_{(r)} + \boldsymbol{M}^{(q)}_{(d)}. \tag{3.2}$$

The exact expressions for the various components in (3.2) are given (see Appendix A) in terms of the Kelvin impulse as

$$\boldsymbol{F}^{(q)}_{(r)} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{M}\boldsymbol{U} + \boldsymbol{Z}\boldsymbol{\Omega}) + \boldsymbol{\Omega} \wedge ((v\rho_b\hat{\mathbf{1}} + \boldsymbol{M})\boldsymbol{U} + \boldsymbol{Z}\boldsymbol{\Omega}), \tag{3.3}$$

$$\boldsymbol{M}^{(q)}_{(r)} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{Z}^T\boldsymbol{U} + \boldsymbol{R}\boldsymbol{\Omega}) + \boldsymbol{\Omega} \wedge (\boldsymbol{Z}^T\boldsymbol{U} + (\boldsymbol{R} + \boldsymbol{I})\boldsymbol{\Omega}) + \boldsymbol{U} \wedge (\boldsymbol{M}\boldsymbol{U} + \boldsymbol{Z}\boldsymbol{\Omega}) \tag{3.4}$$

and

$$\boldsymbol{F}^{(q)}_{(d)} = \frac{\mathrm{d}\boldsymbol{K}(\phi_d)}{\mathrm{d}t} + \boldsymbol{\Omega} \wedge \boldsymbol{K}(\phi_d), \tag{3.5}$$

$$\boldsymbol{M}^{(q)}_{(d)} = \frac{\mathrm{d}\boldsymbol{P}(\phi_d)}{\mathrm{d}t} + \boldsymbol{\Omega} \wedge \boldsymbol{P}(\phi_d) + \boldsymbol{U} \wedge \boldsymbol{K}(\phi_d). \tag{3.6}$$

The Kelvin impulse $\boldsymbol{K}(\phi)$ and the Kelvin-impulse couple $\boldsymbol{P}(\phi)$ for any potential ϕ are defined as

$$\boldsymbol{K}(\phi) \equiv -\int_S \phi\boldsymbol{n}\,\mathrm{d}S, \quad \boldsymbol{P}(\phi) \equiv -\int_S \phi(\boldsymbol{x} \wedge \boldsymbol{n})\,\mathrm{d}S. \tag{3.7}$$

We have also introduced in the above the following 6×6 symmetric added-mass tensor of the body

$$\mathbf{G} \equiv \begin{vmatrix} \boldsymbol{M} & \boldsymbol{Z} \\ \boldsymbol{Z}^T & \boldsymbol{R} \end{vmatrix} \tag{3.8}$$

(Lamb 1945, Ch. 5). Here $\boldsymbol{M}$ denotes the rectilinear, $\boldsymbol{Z}$ is the coupled (linear-angular) and $\boldsymbol{R}$ is the rotational added-mass tensor, defined as

$$M_{ij} \equiv -\int_S \Phi_i n_j\,\mathrm{d}S, \quad Z_{ij} \equiv -\int_S \Psi_j n_i\,\mathrm{d}S, \quad \text{and} \quad R_{ij} \equiv -\int_S \Psi_i(\boldsymbol{x} \wedge \boldsymbol{n})_j\,\mathrm{d}S. \tag{3.9}$$

For future use we define the tensor densities $\boldsymbol{m}(\boldsymbol{x})$ and $\boldsymbol{z}(\boldsymbol{x})$ by

$$m_{ij}(\boldsymbol{x}) \equiv \Phi_i(\boldsymbol{x})n_j(\boldsymbol{x}); \quad z_{ij}(\boldsymbol{x}) \equiv \Psi_i(\boldsymbol{x})n_j(\boldsymbol{x}). \tag{3.10}$$

It is clear that the deformation potential vanishes for a *rigid* body (i.e. $\phi_d = 0$) and thus $\boldsymbol{F}^{(q)}_{(d)} = \boldsymbol{M}^{(q)}_{(d)} = 0$. Equations (3.1) reduce then to the classical Kirchhoff equations given, for example by equation (17.43) of Milne-Thomson (1968).

Let us introduce next the generalized impulses, defined as

$$\boldsymbol{p} \equiv (v\rho_b\hat{\mathbf{1}} + \boldsymbol{M})\boldsymbol{U} + \boldsymbol{Z}\boldsymbol{\Omega} + \boldsymbol{K}(\phi_d), \quad \boldsymbol{l} \equiv \boldsymbol{Z}^T\boldsymbol{U} + (\boldsymbol{I} + \boldsymbol{R})\boldsymbol{\Omega} + \boldsymbol{P}(\phi_d). \tag{3.11}$$

The classical Kirchhoff equations (3.1)–(3.6), which govern the motion of a *deformable* body in a *quiescent* medium, can then be written in terms of $\boldsymbol{p}$ and $\boldsymbol{l}$ as

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} + \boldsymbol{\Omega} \wedge \boldsymbol{p} = 0 \quad \text{and} \quad \frac{\mathrm{d}\boldsymbol{l}}{\mathrm{d}t} + \boldsymbol{\Omega} \wedge \boldsymbol{l} + \boldsymbol{U} \wedge \boldsymbol{p} = 0. \tag{3.12}$$

It is important to note that the generalized impulses introduced above are in fact just the momentum (angular momentum) of the body plus the Kelvin impulse (Kelvin-impulse couple) induced in the fluid due to motion and deformation of the body.

3.2. *The generalized Kirchhoff equations*

It is shown in what follows that the generalized Kirchhoff equations, corresponding to the case of a deformable body moving in an imposed *non-uniform* flow, can be expressed in the following compact form:

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{vmatrix}(\rho_b v U) \\ (I\Omega)\end{vmatrix} + \begin{vmatrix}F^{(q)} \\ M^{(q)}\end{vmatrix} = \begin{vmatrix}A(X,\hat{Q}) & B(X,\hat{Q}) \\ C(X,\hat{Q}) & D(X,\hat{Q})\end{vmatrix}\begin{vmatrix}U \\ \Omega\end{vmatrix} + \begin{vmatrix}F^{(0)}(X,\hat{Q}) \\ M^{(0)}(X,\hat{Q})\end{vmatrix}, \tag{3.13}$$

where the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ depend on the body's geometry and are *linear* vector functionals of the ambient flow field V (evaluated on $S(t)$). The last terms $F^{(0)}$ and $M^{(0)}$ in the right-hand side of (3.13) are *quadratic* vector functionals of V and respectively represent the force and moment acting on a *stationary deformable* body, due to its interactions with the ambient stream V. Clearly, in the absence of an external flow field (i.e. $V = 0$), the right-hand side of (3.13) is equal to zero and (3.13) reduces to the familiar form of (3.1).

Next we prove that the coefficient matrix field in (3.13),

$$\begin{vmatrix}\boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D}\end{vmatrix} \equiv \boldsymbol{W}(X,\hat{\boldsymbol{Q}}), \tag{3.14}$$

is an *antisymmetric* one, i.e.

$$\boldsymbol{W} = -\boldsymbol{W}^T. \tag{3.15}$$

According to Appendices B and C it is first noted that the two-index matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ can defined in terms of the density matrices m_{ik} and z_{ik} given by (3.10) as

$$\boldsymbol{A} = \int_S \left(\boldsymbol{E}\boldsymbol{m}^T - \boldsymbol{m}\boldsymbol{E}\right)\, \mathrm{d}S, \tag{3.16}$$

$$\boldsymbol{B} = \int_S \left(\tau(\boldsymbol{m}V) - \boldsymbol{m}\boldsymbol{V} + \boldsymbol{m}\boldsymbol{E}\boldsymbol{X} + \boldsymbol{E}\boldsymbol{z}^T\right)\, \mathrm{d}S, \tag{3.17}$$

$$\boldsymbol{C} = \int_S \left(\tau(\boldsymbol{m}V) - \boldsymbol{V}\boldsymbol{m} + \boldsymbol{X}\boldsymbol{E}\boldsymbol{m}^T - \boldsymbol{z}\boldsymbol{E}\right)\, \mathrm{d}S, \tag{3.18}$$

where following (2.22)

$$\boldsymbol{V} \equiv \tau(V) \equiv V \wedge (.), \quad \boldsymbol{X} \equiv \tau(\boldsymbol{x}) \equiv \boldsymbol{x} \wedge (.), \quad \text{and} \quad \tau(\boldsymbol{m}V) \equiv (\boldsymbol{m}V) \wedge (.). \tag{3.19}$$

Finally, $\boldsymbol{D}$ is given by

$$\boldsymbol{D} = \boldsymbol{d} \wedge (.), \tag{3.20}$$

where

$$\boldsymbol{d} \equiv \int_S \left(\Psi \wedge \frac{\partial}{\partial n}(\boldsymbol{x} \wedge V) + \boldsymbol{z}V\right)\, \mathrm{d}S. \tag{3.21}$$

Acting on (3.21) with $\partial/\partial n$, one can also express the vector $\boldsymbol{d}$ in an alternative form

$$\boldsymbol{d} = \int_S \left(\left(\boldsymbol{z} + \boldsymbol{z}^T - \mathrm{Tr}(\boldsymbol{z})\right) V + \left(\boldsymbol{E}\boldsymbol{z}^T - \mathrm{Tr}(\boldsymbol{E}\boldsymbol{z}^T)\right)\boldsymbol{x}\right)\, \mathrm{d}S. \tag{3.22}$$

It follows then immediately from (3.16)–(3.20) that

$$\boldsymbol{A} = -\boldsymbol{A}^T, \qquad \boldsymbol{D} = -\boldsymbol{D}^T, \qquad \text{and} \qquad \boldsymbol{C}^T = -\boldsymbol{B}, \tag{3.23}$$

which yields the desired antisymmetry property (3.15). The physical reason for this antisymmetricity results from the fact that a body embedded in an irrotational inviscid stream represents a 'passive' system, i.e. it can only absorb energy but cannot generate it. Thus, the existence of an eigenvalue of $\boldsymbol{W}$ with a non-zero real part (in the case of a rigid body embedded in a stationary flow field) leads to non-conservation of energy (see §6).

It is worth noting that the matrix $\boldsymbol{A}$ depends only on the rate-of-strain tensor $\boldsymbol{E}$ whereas the matrices $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ depend in addition on the velocity field $\boldsymbol{V}$. The antisymmetric tensor $\boldsymbol{A}$ can be also written in the 'magnetic' form as

$$\boldsymbol{A}\boldsymbol{U} = -\left(\nabla_X \wedge \boldsymbol{K}(\acute{\phi}_0)\right) \wedge \boldsymbol{U}, \tag{3.24}$$

where

$$\boldsymbol{K}(\acute{\phi}_0) = -\int_S \boldsymbol{n}\acute{\phi}_0 \, \mathrm{d}S = \int_S \boldsymbol{\Phi}(\boldsymbol{n} \cdot \boldsymbol{V}) \, \mathrm{d}S, \tag{3.25}$$

is the Kelvin impulse of the fluid motion induced as a result of introducing a stationary body into the ambient stream $\boldsymbol{V}$. The gauge invariance of $\boldsymbol{K}(\acute{\phi}_0)$ is already accounted for, due to the fact that $\boldsymbol{K}(\acute{\phi}_0)$ is solenoidal, i.e.

$$\nabla_X \cdot \boldsymbol{K}(\acute{\phi}_0) = 0. \tag{3.26}$$

One can also see that the body's shape deformations contribute to (3.16)–(3.20) only through the time-dependence of S and the Kirchhoff potentials $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$. Thus, there is basically no coupling between the rigid body motion and the body's deformation.

The force components $\boldsymbol{A}\boldsymbol{U}$ and $\boldsymbol{D}\boldsymbol{\Omega}$ are orthogonal to $\boldsymbol{U}$ and $\boldsymbol{\Omega}$ respectively and therefore, are of a 'lift-like' type. On the other hand, the forces $\boldsymbol{B}\boldsymbol{\Omega}$ and $\boldsymbol{C}\boldsymbol{U}$ have components both in the orthogonal direction and in the directions of $\boldsymbol{\Omega}$ and $\boldsymbol{U}$. Such forces induce, in general, a spiralling motion of the body (similar to the precession of a charged particle in a magnetic field, see for example Landau & Lifshitz 1989, §21). To illustrate this effect let us consider the rectilinear motion (i.e. $\boldsymbol{\Omega} = 0$) where $|\boldsymbol{U}| \gg |\boldsymbol{V}|$ (the so-called 'fast body' assumption). In this case the force $\boldsymbol{F}^{(0)}$ (quadratically depending on $\boldsymbol{V}$) can be neglected in comparison with the force $\boldsymbol{A}\boldsymbol{U}$ (linearly depending on $\boldsymbol{V}$). The governing equation (3.13), reduces now to the following form:

$$(v\rho_b\hat{\mathbf{1}} + \boldsymbol{M})\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} = \boldsymbol{A}\boldsymbol{U}. \tag{3.27}$$

For a constant $\boldsymbol{A}$, the corresponding motion of the body is a rectilinear one in the direction of the vector $\tau^{-1}(\boldsymbol{A})$, combined with a rotation in the plane orthogonal to $\tau^{-1}(\boldsymbol{A})$ imposed on an elliptical trajectory. The resulting spiralling motion is related to the work of Saffman (1956, p. 253) (see also Benjamin 1987, Ch. 5), who claimed 'a bubble could usually be made to spiral...by placing an obstacle ...in the path of the bubble'. A possible explanation for this phenomena is that the presence of an obstacle in a stream generally leads to some sort of flow field non-uniformity. In turn, it results in an additional non-zero force $\boldsymbol{A}\boldsymbol{U}$ which acts on the bubble.

3.3. *Force and moment acting on a stationary body*

All terms in (2.11), (2.12), (2.18) and (2.19) which depend on $\boldsymbol{V}$ but are independent of $\boldsymbol{U}$ and $\boldsymbol{\Omega}$, should be incorporated into the expression for the force $\boldsymbol{F}^{(0)}$ and moment $\boldsymbol{M}^{(0)}$ acting on a stationary body embedded in a non-uniform flow field $\boldsymbol{V}$. These can also be written in a similar manner to (3.2) as

$$\boldsymbol{F}^{(0)} \equiv \boldsymbol{F}^{(0)}_{(r)} + \boldsymbol{F}^{(0)}_{(d)}, \qquad \text{and} \qquad \boldsymbol{M}^{(0)} \equiv \boldsymbol{M}^{(0)}_{(r)} + \boldsymbol{M}^{(0)}_{(d)}, \tag{3.28}$$

where the subscripts (r) and (d) denote, as before, rigid and deformable contributions respectively. The force $\boldsymbol{F}^{(0)}_{(r)}$ is given in accordance with (2.11) and (2.18) by

$$\boldsymbol{F}^{(0)}_{(r)} = \frac{\partial}{\partial t}\int_S (\phi + \acute{\phi})\boldsymbol{n} \, \mathrm{d}S + \int_S \left(\acute{\phi}_0 \frac{\partial \boldsymbol{V}}{\partial n} - \frac{\partial \acute{\phi}_0}{\partial n}\boldsymbol{V}\right) \, \mathrm{d}S. \tag{3.29}$$

Taking then into account that (for a stationary body)

$$-\int_S \frac{\partial \acute{\phi}_0}{\partial n} \boldsymbol{V}\, \mathrm{d}S = \int_v \boldsymbol{E}\boldsymbol{V}\mathrm{d}v, \quad \text{and} \quad \int_S \acute{\phi}\boldsymbol{n}\, \mathrm{d}S = -\int_S \boldsymbol{\Phi n} \cdot \boldsymbol{V}\, \mathrm{d}S, \tag{3.30}$$

we finally obtain by using the Gauss theorem

$$\boldsymbol{F}^{(0)}_{(r)} = \int_S \left(\acute{\phi}_0 \frac{\partial \boldsymbol{V}}{\partial n} - \boldsymbol{\Phi n} \cdot \frac{\partial \boldsymbol{V}}{\partial t} \right) \mathrm{d}S + \int_v \frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t} dv, \tag{3.31}$$

where we have introduced in the above the substantional derivative symbol, $\mathrm{D}/\mathrm{D}t \equiv \partial/\partial t + \boldsymbol{V} \cdot \nabla$ (denoting the liquid acceleration measured in the moving coordinate system). Equation (3.31) is not more than a variant of the statement

$$\boldsymbol{F}^{(0)}_{(r)} = -\int_S (p + \acute{p})\boldsymbol{n}\, \mathrm{d}S, \tag{3.32}$$

where p is the ambient pressure and $\acute{p}$ is the additional pressure due to the instantaneous introduction of the body into the ambient stream.

A similar expression is obtained for the moment $\boldsymbol{M}^{(0)}_{(r)}$, by using (2.12) and (2.19), i.e.

$$\boldsymbol{M}^{(0)}_{r} = \left(\int_S \acute{\phi}_0 \frac{\partial(\boldsymbol{x} \wedge \boldsymbol{V})}{\partial n} \mathrm{d}S - \int_S \boldsymbol{\Psi n} \cdot \frac{\partial \boldsymbol{V}}{\partial t} \mathrm{d}S \right) + \int_v \boldsymbol{x} \wedge \frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t} \mathrm{d}v. \tag{3.33}$$

The deformation parts of the force $\boldsymbol{F}^{(0)}_{(d)}$ and moment $\boldsymbol{M}^{(0)}_{d}$, which arise due to the interaction of the deformation potential ϕ_d with the ambient flow field, are further obtained as

$$\boldsymbol{F}^{(0)}_{(d)} = \frac{\delta}{\delta t} \left(\int_v \boldsymbol{V}\, \mathrm{d}v - \int_S \boldsymbol{\Phi n} \cdot \boldsymbol{V}\, \mathrm{d}S \right) + \left(\int_S \phi_d \frac{\partial \boldsymbol{V}}{\partial n}\, \mathrm{d}S - \int_S \frac{\partial \phi_d}{\partial n} \boldsymbol{V}\, \mathrm{d}S \right). \tag{3.34}$$

The time differential operator $\delta/\delta t$ acts here only on time-dependent variables resulting from pure deformations.

Recalling (2.1) and (2.8) and using the Transport theorem, lead to

$$\frac{\delta}{\delta t} \int_v \boldsymbol{V}\, \mathrm{d}v = -\int_S \frac{\dot{S}}{|\nabla S|} \boldsymbol{V}\, \mathrm{d}S; \quad -\int_S \frac{\partial \phi_d}{\partial n} \boldsymbol{V}\, \mathrm{d}S = \int_S \frac{\dot{S}}{|\nabla S|} \boldsymbol{V}\, \mathrm{d}S, \tag{3.35}$$

from which one finally obtains

$$\boldsymbol{F}^{(0)}_{(d)} = \int_S \phi_d \frac{\partial \boldsymbol{V}}{\partial n} \mathrm{d}S - \frac{\delta}{\delta t} \int_S \boldsymbol{\Phi n} \cdot \boldsymbol{V} \mathrm{d}S. \tag{3.36}$$

A similar expression can be derived for $\boldsymbol{M}^{(0)}_{(d)}$:

$$\boldsymbol{M}^{(0)}_{(d)} \equiv \int_S \phi_d \frac{\partial(\boldsymbol{x} \wedge \boldsymbol{V})}{\partial n} \mathrm{d}S - \frac{\delta}{\delta t} \int_S \boldsymbol{\Psi n} \cdot \boldsymbol{V}\, \mathrm{d}S. \tag{3.37}$$

Clearly, for a rigid body $\phi_d = 0$ and $\delta/\delta t = 0$, which implies that in this case $\boldsymbol{F}^{(0)}_{(d)} = 0$ and $\boldsymbol{M}^{(0)}_{d} = 0$.

The force $\boldsymbol{F}^{(0)}$ is in general not of a potential type, namely,

$$\nabla \wedge \boldsymbol{F}^{(0)} = -\frac{\partial(\nabla \wedge \boldsymbol{K}(\acute{\phi}_0))}{\partial t}. \tag{3.38}$$

Therefore for a stationary ambient flow field the force $\boldsymbol{F}^{(0)}_{(r)}$ is a potential one and

$$\boldsymbol{F}^{(0)}_{(r)} = -\nabla_X \pi(\boldsymbol{X}, \boldsymbol{Q}), \tag{3.39}$$

where the effective potential is given by

$$\pi = -\frac{1}{2}\int_v V^2 \mathrm{d}v + \frac{1}{2}\int_S \phi_0' \frac{\partial \phi_0'}{\partial n}\mathrm{d}S. \tag{3.40}$$

It follows then from (3.40) that $\pi \leqslant 0$, since the last integral is always negative.

4. Forces in a weakly non-uniform flow field

In many practical cases it is justifiable to assume that the characteristic length scale of the non-uniformity of the ambient flow field V is much larger than the characteristic length scale d of the body (the so called 'weak-straining' field approximation). As a direct consequence of this weakly non-uniform flow assumption, one can keep only the first two terms in a Taylor expansion of V about the body centroid X, namely

$$V(X+x,t) = V(X,t) + \mathbf{E}(X,t)x + O(\epsilon^2). \tag{4.1}$$

There exists therefore a small parameter ϵ, which can be expressed in terms of the strain tensor $\mathbf{E}$, as

$$\epsilon = \frac{\|\nabla \mathbf{E}\| d}{\|\mathbf{E}\|}, \tag{4.2}$$

where $\|(.)\|$ denotes the norm of the tensor

The substitution of (4.1) in (3.16) for $\mathbf{A}$ and in (3.17) for $\mathbf{B}$, gives

$$\mathbf{A}U = (\mathbf{ME} - \mathbf{EM})U, \tag{4.3}$$

and

$$\mathbf{B}\Omega = \mathbf{M}(V_0 \wedge \Omega) - \mathbf{M}V_0 \wedge \Omega - \mathbf{EZ}\Omega + \tfrac{1}{2}(\mathbf{S}{:}\mathbf{E}) \wedge \Omega - \frac{1}{2}\mathbf{S} : (\mathbf{E}\hat{\Omega} - \hat{\Omega}\mathbf{E}), \tag{4.4}$$

where we define the third-order tensor $\mathbf{S}$ (a purely geometrical parameter depending on the shape of the body), as

$$\mathbf{S}_{ijk} \equiv \int_S \Phi_i \frac{\partial (x_j x_k)}{\partial n}\mathrm{d}S. \tag{4.5}$$

The convolutions in (4.4) are denoted by $(\mathbf{S} : \mathbf{E})_i \equiv \mathbf{S}_{ijk}E_{jk}$ and $(\mathbf{S} : (\mathbf{E}\hat{\Omega} - \hat{\Omega}\mathbf{E}))_i \equiv \mathbf{S}_{ijk}(E_{jp}\Omega_{pk} - \Omega_{jp}E_{pk})$. Also recall that according to (2.22) $\Omega_{pk} = -\epsilon_{pkq}\Omega_q$.

What remains now to do is to evaluate the free force $F^{(0)}$ (3.28), for which one can use (3.31) for the 'rigid' part $F^{(0)}_{(r)}$, resulting in

$$\int_S \phi_0' \mathbf{E}n\,\mathrm{d}S + \int_v \mathbf{E}V\mathrm{d}v = \mathbf{E}\int_S \phi_0' n\,\mathrm{d}S + \mathbf{E}\int_v V\mathrm{d}v + O(\epsilon^2)$$

$$= -\mathbf{E}\int_S \Phi V\cdot n\,\mathrm{d}S + v\mathbf{E}V_0 + O(\epsilon^2) = \mathbf{E}(\mathbf{M} + v\hat{\mathbf{1}})V_0 + O(\epsilon^2). \tag{4.6}$$

In addition, one obtains from (3.31)

$$\int_v \frac{\partial V}{\partial t}\mathrm{d}v - \int_S \Phi n \cdot \frac{\partial V}{\partial t}\mathrm{d}S = (\mathbf{M} + v\hat{\mathbf{1}})\frac{\partial V}{\partial t} - \frac{1}{2}\mathbf{S} : \frac{\partial \mathbf{E}}{\partial t} + O(\epsilon^2), \tag{4.7}$$

and correspondingly for the deformation part of the force $F^{(0)}_{(d)}$ (3.36),

$$\int_S \phi_d \frac{\partial V}{\partial n}\mathrm{d}S - \frac{\delta}{\delta t}\int_S \Phi n \cdot V\mathrm{d}S = \mathbf{E}K(\phi_d) + \frac{\delta \mathbf{M}}{\delta t}V_0 - \frac{1}{2}\frac{\delta \mathbf{S}}{\delta t} : \mathbf{E} + O(\epsilon^2). \tag{4.8}$$

Equations (4.3) to (4.8) are identical with the corresponding expressions, recently derived by Galper & Miloh (1994, eqs. (3.25) and (3.26)) for the force acting on a deformable body moving in a weakly non-uniform ambient flow field. Thus, the expression for the part $\boldsymbol{F}^{(0)}_{(r),e}$ of the force $\boldsymbol{F}^{(0)}_{(r)}$ which arises due to the interactions between the rigid body's motion and the non-uniformity of the imposed stationary stream, takes the form

$$\boldsymbol{F}^{(0)}_{(r),e} = (\boldsymbol{ME}-\boldsymbol{EM})\boldsymbol{U} + \boldsymbol{E}(\boldsymbol{M}\boldsymbol{V}_0 + v\boldsymbol{V}_0 - \boldsymbol{Z\Omega}) - \tfrac{1}{2}\boldsymbol{\Omega}\wedge(\boldsymbol{S}:\boldsymbol{E}) - \tfrac{1}{2}(\boldsymbol{S}:(\boldsymbol{E}\hat{\boldsymbol{\Omega}} - \hat{\boldsymbol{\Omega}}\boldsymbol{E})). \quad (4.9)$$

Substituting (4.1) in (3.33) and (3.37), we rederived in a similar manner the corresponding expression for the moment acting on a body placed in a weakly non-uniform flow field, recently obtained by Galper & Miloh (1994, eqs. (4.29)–(4.31)).

For the particular case where the ambient flow field is *linear*, i.e.

$$\boldsymbol{V}(\boldsymbol{X},t) = \boldsymbol{V}_0(t) + \boldsymbol{E}(t)\boldsymbol{X}, \quad (4.10)$$

(4.9) is an *exact* statement except for an additional term (arising from (3.31)), namely

$$-\tfrac{1}{2}\boldsymbol{E}(\boldsymbol{S}:\boldsymbol{E}). \quad (4.11)$$

For a rigid body with three mutually orthogonal planes of symmetry, one gets $\boldsymbol{Z}=0$ and $\boldsymbol{S}=0$. Thus, at least within the realm of the weakly non-uniform flow assumption, the angular velocity $\hat{\boldsymbol{\Omega}}$ does not directly interact with the rate-of-strain tensor $\boldsymbol{E}$.

5. Applications to bubble dynamics

5.1. *Impulsive motion*

In some problems of bubble dynamics it is of interest to evaluate the velocity of the bubble resulting from an impulsive motion of the surrounding liquid with an instantaneous velocity distribution $\boldsymbol{V}(\boldsymbol{x})$ (Wijngaarden 1976). In order to determine the impulsive nature of the body's motion, we integrate (3.13), with the right-hand side given by (3.16) to (3.20) and (3.31) to (3.33), from $t=0^-$ to $t=0^+$ (i.e. at the instant when the liquid has been set impulsively into motion). By keeping only terms with a partial time-derivatives one can then derive the following set of equations which are in fact equivalent to a statement concerning the conservation of both the impulse $\boldsymbol{p}_b = \boldsymbol{K}(\phi+\acute{\phi})$ and the impulse couple, namely $\boldsymbol{l}_b = \boldsymbol{P}(\phi+\acute{\phi})$

$$(v\rho_b\hat{\mathbf{1}} + \boldsymbol{M})\boldsymbol{U} + \boldsymbol{Z\Omega} = \int_v \boldsymbol{V}\mathrm{d}v - \int_S \boldsymbol{\Phi n}\cdot\boldsymbol{V}\mathrm{d}S, \quad (5.1)$$

and

$$(\boldsymbol{I}+\boldsymbol{R})\boldsymbol{\Omega} + \boldsymbol{Z}^T\boldsymbol{U} = \int_v \boldsymbol{x}\wedge\boldsymbol{V}\mathrm{d}v - \int_S \boldsymbol{\Psi n}\cdot\boldsymbol{V}\mathrm{d}S. \quad (5.2)$$

For ellipsoidal shapes the Kirchhoff potential $\boldsymbol{\Phi}$ may be expressed according to Lamb (1945, Ch. 5) as

$$\boldsymbol{\Phi}\Big|_S = -\frac{1}{v}\boldsymbol{Mx}\Big|_S. \quad (5.3)$$

Using (5.3), one gets

$$(v\rho_b\hat{\mathbf{1}} + \boldsymbol{M})\boldsymbol{U} = (\hat{\mathbf{1}} + \frac{\boldsymbol{M}}{v})\int_v \boldsymbol{V}\mathrm{d}v. \quad (5.4)$$

For a spherical bubble ($\rho_b = 0$), it immediately leads to Wijngaarden's (1976) result

$$U = 3V_c, \tag{5.5}$$

where V_c denotes the value of the velocity field at the sphere's centre.

Since among the set of all smooth convex three-dimensional bodies with a prescribed volume, the spherical shape has the minimum trace of the added-mass tensor (e.g. Shiffer 1975), we may conclude that the minimum velocity acquired by the bubble due to the impulsive motion of the ambient fluid, is given by $U = (3/v)\int_v V \mathrm{d}v$. This velocity is precisely attained for a *spherical* shape. According to (5.2), a non-spherical body may acquire in addition an angular velocity.

5.2. *Self-propulsion in a non-uniform flow field*

Let us next discuss the intriguing effect of self-propulsion of a deformable body in a perfect fluid. It is well known that periodic surface deformations can lead to a self-induced persistent motion of the body. Here we consider the qualitative differences between self-propulsion effects of a deformable body which is placed in an otherwise *quiescent* fluid (Miloh & Galper 1993) and in a *non-uniform* ambient flow field (Galper & Miloh 1995). As demonstrated below, the self-propulsion mechanism in these two cases is essentially different.

As a simple example, we analyse the case of a periodic instantaneous shape change from a surface $S_1(\boldsymbol{x})$ say at $t = 2n\tau^-$, to a surface $S_2(\boldsymbol{x})$ at $t = (2n+1)\tau^-$, where 2τ is the period and $n = 0, 1, \ldots$. Since such an instantaneous change in $S(\boldsymbol{x})$ occurs at $t = m\tau$, $m = 0, 1, 2\ldots$, we define $\tau^{\mp} \equiv \lim_{\nu\to 0}(\tau \mp \nu)$. Our interest lies here mainly in the case where $\tau \to 0$. For a *quiescent* ambient flow it is evident that such an impulsive deformation cannot produce any velocity of self-propulsion, due to the time reversibility nature of the deformation (see Childress 1981, Ch. 8 and Benjamin & Ellis 1990). It is also implied that the change in the body's velocity during the deformation period is principally governed by the impulsive deformation force $\boldsymbol{F}_d^{(0)}$ (3.36), i.e.

$$\left| \int_0^{2\tau} \boldsymbol{F}_{(r)}^{(0)} \, \mathrm{d}t \right| \ll \left| \int_0^{2\tau} \boldsymbol{F}_{(d)}^{(0)} \, \mathrm{d}t \right|. \tag{5.6}$$

This inequality can always be satisfied by choosing a small enough τ.

The shapes S_1 and S_2 must be simply connected so as to exclude the possibility of vortex shedding (as it is, for example, in the case of toroidal bubbles, e.g. Best 1993). The total rate of change of the deformation Kelvin impulse $\int_S \phi_d \boldsymbol{n} \, \mathrm{d}S$ during a period is obviously zero. To demonstrate the basic differences between the self-propulsion mechanism in uniform and non-uniform flow fields, let us examine the simple case where S_1 and S_2 are both spherical with radii a_1 and a_2, respectively. By enforcing (5.6), one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_b + \tfrac{1}{2})v\boldsymbol{U} = \boldsymbol{F}_{(d)}^{(0)} = \tfrac{1}{2}\dot{v}\boldsymbol{V}(\boldsymbol{X}), \tag{5.7}$$

where $\dot{v}$ denotes the rate of volume change. In obtaining the last term in the right-hand side of (5.7), representing the effect of non-isochoric deformation, we use (3.36) which for a spherical shape gives

$$\int_S \phi_d \frac{\partial}{\partial n} \boldsymbol{V} \mathrm{d}S = 0, \qquad \frac{\delta}{\delta t}\int_S \boldsymbol{\Phi n} \cdot \boldsymbol{V} \mathrm{d}S = -\frac{1}{2}\frac{\delta}{\delta t}\int_v \boldsymbol{V} \mathrm{d}v = -\frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{V}(\boldsymbol{X}). \tag{5.8}$$

Thus, for a pulsating sphere with a time-dependent radius, which is embedded in a non-uniform flow field, the deformation part of the force $\boldsymbol{F}_{(d)}^{(0)}$ depends locally on the ambient flow field $\boldsymbol{V}$. Furthermore, integrating (5.7) across the discontinuity from $t = 0^-$ to $t = \tau^-$, one gets

$$(\rho_b + \tfrac{1}{2})(v_2 \boldsymbol{U}_2 - v_1 \boldsymbol{U}_1) = \tfrac{1}{2}(v_1 - v_2)\boldsymbol{V}(\boldsymbol{X}). \tag{5.9}$$

Note that in (5.9) $\boldsymbol{U}_1 \equiv \boldsymbol{U}(0^-)$, $\boldsymbol{U}_2 \equiv \boldsymbol{U}(0^+)$, $\boldsymbol{U}_3 \equiv \boldsymbol{U}(\tau^+)$ and v_1, v_2 denote the volumes at $t = 0^-$ and $t = 0^+$ respectively. It follows then from (5.6) that the sphere propels itself with a constant speed from 0^+ to τ^- and thus $\boldsymbol{U}(\tau^-) = \boldsymbol{U}_2$.

Integrating next (5.7) from τ^- to $2\tau^-$, yields

$$(\rho_b + \tfrac{1}{2})(v_1 \boldsymbol{U}_3 - v_2 \boldsymbol{U}_2) = \tfrac{1}{2}(v_2 - v_1)\boldsymbol{V}(\boldsymbol{X} + \tau(\boldsymbol{U}_2 + \boldsymbol{U}_3)). \tag{5.10}$$

Finally, by letting $\tau \to 0$ and using both (5.9) and (5.10), one obtains the following ordinary differential equation which governs the sphere's motion:

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{U}, \qquad \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} = \gamma \boldsymbol{E}(\boldsymbol{X})\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} + \alpha \boldsymbol{E}(\boldsymbol{X})\boldsymbol{V}(\boldsymbol{X}) + O\left(\left(\frac{|\boldsymbol{U}_1|\tau}{D}\right)^2\right), \tag{5.11}$$

with the constants

$$\gamma \equiv \frac{(v_2^2 - v_1^2)}{v_1 v_2 (2\rho_b + 1)} \quad \text{and} \quad \alpha \equiv \frac{(v_2 - v_1)^2}{v_1 v_2 (2\rho_b + 1)}.$$

Here D denotes the characteristic length scale of the flow non-uniformity and the initial condition for (5.11) is readily given by

$$\boldsymbol{U}|_{t=0} = \boldsymbol{U}_1. \tag{5.12}$$

Assuming in addition that

$$|\boldsymbol{V}|(\boldsymbol{X}) \ll |\boldsymbol{U}_1| \tag{5.13}$$

(i.e. a 'fast motion' relative to the ambient flow), one can neglect the last term in the right-hand side of (5.11) and reduce it to

$$\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} = \gamma \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}. \tag{5.14}$$

The latter can be also integrated to give

$$\boldsymbol{U} = \boldsymbol{U}_1 + \gamma(\boldsymbol{V}(\boldsymbol{X}(t)) - \boldsymbol{V}(0)). \tag{5.15}$$

For the purpose of illustration, let us next choose a particular case where the ambient flow field $\boldsymbol{V}$ depends only on a *single* Cartesian coordinate coinciding with the direction of $\boldsymbol{U}_1$ (say, the y-axis with a corresponding z-axis in the orthogonal direction). Then, by enforcing (5.13) and integrating (5.14) one obtains,

$$\langle \boldsymbol{U}_z \rangle = \gamma \left(\langle \boldsymbol{V}_z \left(|\boldsymbol{U}_1| t \right) \rangle - \boldsymbol{V}_z(0) \right), \tag{5.16}$$

where $\boldsymbol{V} \equiv (V_x, V_y, V_z)$ and $\boldsymbol{U} \equiv (U_x, U_y, U_z)$. If $\boldsymbol{V}_z$ is a space-periodic (stationary) flow field, then $\langle \boldsymbol{V}_z \left(|\boldsymbol{U}_1| t \right) \rangle = 0$, and thus

$$\langle \boldsymbol{U}_z \rangle = -\gamma \left(\boldsymbol{V}_z(0) \right). \tag{5.17}$$

Hence, except for a motion with a constant velocity $\boldsymbol{U}_1$ (the case of a uniform flow), there arises, in addition, a persistent (non-zero) velocity of self-propulsion, given by $-\gamma \boldsymbol{V}(0)$, in the direction orthogonal to $\boldsymbol{U}_1$. This extra term results from *nonlinear* parametric interactions between the body's volume deformation and the non-uniformity of the stationary ambient flow. This newly found nonlinear mechanism of parametric resonance self-propulsion, is different in essence from the direct self-propulsive mechanism. The later arises from *linear* resonant interactions between volume- and surface-deformation modes in a quiescent flow field (Miloh & Galper 1993), or between body's deformations and non-uniform *time-dependent* ambient flow field (Galper & Miloh 1995). The case of periodic *isochoric* (volume-preserving) surface deformations of a bubble which is embedded in a *stationary* arbitrary non-uniform flow field has been recently treated by Galper & Miloh (1995). It is demonstrated there that the resonant frequency of the surface deformation is proportional to the magnitude of the flow non-uniformity.

6. The first integral

An important result which follows directly from the dynamical system (3.13), is that the latter always has a first integral for the case of a *rigid* body moving in a *stationary* flow field. This conclusion can be also considered as a generalization of the traditional familiar Kirchhoff integral for the motion of a rigid body in an otherwise quiescent fluid, i.e.

$$\tfrac{1}{2}(\rho_b v\hat{\mathbf{1}} + \boldsymbol{M})\boldsymbol{U}\cdot\boldsymbol{U} + \tfrac{1}{2}(\boldsymbol{I}+\boldsymbol{R})\boldsymbol{\Omega}\cdot\boldsymbol{\Omega} + \boldsymbol{Z}\boldsymbol{U}\cdot\boldsymbol{\Omega} = \text{const.} \tag{6.1}$$

In addition there exist two other first integrals (which also hold for a deformable body),

$$\boldsymbol{p}^2 = \text{const}, \quad \text{and} \quad \boldsymbol{p}\cdot\boldsymbol{l} = \text{const}, \tag{6.2}$$

where the generalized impulses are defined in (3.11)). The last two conservation laws reflect the properties of translational and reflectional invariance for the quiescent case and in general they cannot be generalized for an arbitrary ambient flow field.

The motion of a solid body in a surrounding fluid, which is otherwise at rest, is known to be Hamiltonian (Novikov 1981) and for this reason (6.1) is indeed a manifestation of the energy conservation principle. In order to generalize the first integral (6.1) for the case of a *non-uniform* ambient flow field, we multiply both sides of (3.13) by $\begin{vmatrix}\boldsymbol{U}\\ \boldsymbol{\Omega}\end{vmatrix}$ and notice first

$$\boldsymbol{F}^{(q)}\cdot\boldsymbol{U} + \boldsymbol{M}^{(q)}\cdot\boldsymbol{\Omega} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{M}\boldsymbol{U}\cdot\boldsymbol{U} + \boldsymbol{R}\boldsymbol{\Omega}\cdot\boldsymbol{\Omega} + 2\boldsymbol{Z}\boldsymbol{U}\cdot\boldsymbol{\Omega}). \tag{6.3}$$

Furthermore, by using the antisymmetry property (3.16), one finds that

$$\boldsymbol{W}(\boldsymbol{U},\boldsymbol{\Omega})^T\cdot(\boldsymbol{U},\boldsymbol{\Omega})^T = 0. \tag{6.4}$$

The next step is to show that the sum $\boldsymbol{F}^{(0)}\cdot\boldsymbol{U} + \boldsymbol{M}^{(0)}\cdot\boldsymbol{\Omega}$ can be written as a full time derivative. This can be done by substituting (3.31) for $\boldsymbol{F}^{(0)}_{(r)}$ and (3.33) for $\boldsymbol{M}^{(0)}_{(r)}$, which by virtue of (2.9) lead to

$$\boldsymbol{U}\cdot\int_S \phi_0'\frac{\partial \boldsymbol{V}}{\partial n}\mathrm{d}S + \boldsymbol{\Omega}\cdot\int_S \phi_0'\frac{\partial(\boldsymbol{x}\wedge\boldsymbol{V})}{\partial n}\mathrm{d}S = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_S \phi_0'\frac{\partial\phi_0'}{\partial n}\mathrm{d}S. \tag{6.5}$$

In deriving (6.5) we have used the definition (2.25) for the vector field $\boldsymbol{V}(\boldsymbol{x})$ and the symmetric property of the Green function $G^{(out)}(\boldsymbol{x},\boldsymbol{y})$. Correspondingly, the Euler equation $\mathrm{D}/\mathrm{D}t\,\boldsymbol{V} = -\nabla p$ (p being the pressure) combined with the Bernoulli equation, implies that

$$\boldsymbol{U}\cdot\int_v \frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t}\mathrm{d}v + \boldsymbol{\Omega}\cdot\int_v \boldsymbol{x}\wedge\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t}\mathrm{d}v = -\frac{\mathrm{d}}{\mathrm{d}t}\int_v p\,\mathrm{d}v = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_v V^2\mathrm{d}v. \tag{6.6}$$

Combining then (6.3) to (6.6) with the dynamical equation (3.13), yields

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\tfrac{1}{2}(\rho_b v\hat{\mathbf{1}}+\boldsymbol{M})\boldsymbol{U}\cdot\boldsymbol{U}+\tfrac{1}{2}(\boldsymbol{I}+\boldsymbol{R})\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}+\boldsymbol{Z}\boldsymbol{U}\cdot\boldsymbol{\Omega}\right)-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_v \boldsymbol{V}^2\mathrm{d}v + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_S \phi_0'\frac{\partial\phi_0'}{\partial n}\mathrm{d}S = 0, \quad (6.7)$$

which, upon time-integration finally gives

$$\tfrac{1}{2}(\rho_b v\hat{\mathbf{1}}+\boldsymbol{M})\boldsymbol{U}\cdot\boldsymbol{U}+\tfrac{1}{2}(\boldsymbol{I}+\boldsymbol{R})\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}+\boldsymbol{Z}\boldsymbol{U}\cdot\boldsymbol{\Omega}-\frac{1}{2}\int_v \boldsymbol{V}^2\mathrm{d}v+\frac{1}{2}\int_S \phi_0'\frac{\partial\phi_0'}{\partial n}\mathrm{d}S = \text{const.} \quad (6.8)$$

In the absence of any imposed flow field $\boldsymbol{V}$, $\phi_0' = 0$ and (6.8) reduces to the classical Kirchhoff integral governing the motion of a rigid body in an otherwise quiescent fluid of infinite expanse. The last two additional terms in (6.8) have the physical meaning of work done against the pressure in placing the body into the ambient flow field at the point $\boldsymbol{X}$. The term $-\frac{1}{2}\int_S \phi_0'(\partial\phi_0'/\partial n)\mathrm{d}S$ can be also interpreted as the kinetic energy of the surrounding flow field, induced by introducing a *stationary* body into the fluid. Similarly, the integral $\frac{1}{2}\int_v \boldsymbol{V}^2\,\mathrm{d}v$ represents the kinetic energy of the fluid motion within the volume v, when the body is absent. Thus, the first integral (6.8) is in fact a variant of the energy conservation principle, which can also be stated as:

The kinetic energy of the rigid body plus *the kinetic energy of the response motion of the fluid (due to the body's motion), expressed in terms of the body's added-mass tensor* minus *the kinetic energy of the ambient flow field within v* minus *the kinetic energy of the response motion of the fluid (resulting from introducing the* stationary *body into the fluid)* equals a *constant.*

The first integral (6.8) clearly reveals the time-symmetry property of the present dynamical system, where the time enters into the formulation (for a *stationary* ambient flow field) only as a parameter in the boundary conditions. It is expressed in terms of the body's coordinates and its geometrical parameters on one hand and the values of the prescribed ambient field $\boldsymbol{V}$ evaluated on the body's surface S on the other. Written in the form (6.8), the first integral hinges upon a time translation symmetry of the system, which suggests that this first integral can be chosen as the Hamiltonian governing the body's motion expressed in terms of the appropriate conjugated variables.

7. Hamiltonian formalism

In this section we prove for the first time the existence of the Hamiltonian approach to the motion of a *deformable* body in an *arbitrary* flow field. A formal Hamiltonian representation of the motion of a *rigid* body in a *quiescent* fluid, was first demonstrated by Lamb (1945, §132) and in terms of the modern framework of the Lie–Poisson brackets by Novikov (1981).

The corresponding Hamiltonian is given by

$$H^{(q)} = \frac{1}{2}\begin{vmatrix}\boldsymbol{p}\\ \boldsymbol{l}\end{vmatrix}\cdot\boldsymbol{J}^{-1}\begin{vmatrix}\boldsymbol{p}\\ \boldsymbol{l}\end{vmatrix}, \quad (7.1)$$

where the total mass matrix $\boldsymbol{J}$ is defined as

$$\boldsymbol{J} \equiv \begin{vmatrix}\rho v\hat{\mathbf{1}}+\boldsymbol{M} & \boldsymbol{Z}\\ \boldsymbol{Z}^T & \boldsymbol{I}+\boldsymbol{R}\end{vmatrix}, \quad (7.2)$$

and the generalized impulse $\boldsymbol{p}$ and impulse couple $\boldsymbol{l}$ are given by (3.11) with $\phi_d = 0$. As general coordinates for the body we choose the six coordinates $\boldsymbol{X}$ and $\mathbf{Q}$, with the conjugated moments $\boldsymbol{p}$ and $\boldsymbol{l}$ respectively. The standard Lie–Poisson brackets for the quiescent case, which correspond to the geometry of the phase space of the system, are degenerated due to the high symmetry of the system. Only after restriction on the surface of the level of the two first integrals (6.2), does the system admit introduction of the canonical Poisson brackets (Novikov 1981).

The advantages of using Hamiltonian formalism for example for analysing bubble dynamics in an unbounded flow field otherwise at rest, have been recently pointed out by Benjamin (1987). Such a representation enables one to account for the conservations laws of bubble dynamics in a systematic way. Nevertheless, there exists a basic difference between Benjamin's Hamiltonian approach and the present one. The Hamiltonian suggested by Benjamin (1987, eq. (2.7)) governs the combined motion of a non-deformable bubble *together* with the surrounding fluid (including the bubble free surface). In contrast, in the present formalism the Hamiltonian for the body includes variables connected *only* to the body's motion. The influence of the ambient flow field is accounted for through a finite number of ambient parameters, functionally depending on the prescribed motion of the fluid (without the body). In other words, the problem of fluid motion in response to the body's motion is considered as resolved within the proposed framework. The corresponding Hamiltonian governs *only* the body's dynamics and is expressed in a body-attached coordinate system. Thus, it is indeed remarkable that the present fluid–body interaction problem, is reduced to a *finite*-dimension dynamical system, which still remains Hamiltonian.

In order to express the equations of motion (3.13) for a deformable body in a Hamiltonian form, let us consider first the case of a translational motion without rotation (i.e. $\boldsymbol{\Omega} = 0$, $\mathbf{Q} = \hat{\mathbf{1}}$). After establishing the particular form (3.24)–(3.26), one can construct the full Lagrangian for a non-rotating body as

$$L(\boldsymbol{X};\boldsymbol{U}) = \tfrac{1}{2}\boldsymbol{J}\boldsymbol{U}\cdot\boldsymbol{U} + \boldsymbol{K}(\phi_0' + \phi_d)\cdot\boldsymbol{U} - \pi(\boldsymbol{X},\mathbf{Q}=\hat{\mathbf{1}}), \tag{7.3}$$

where the potential $\pi(\boldsymbol{X},\mathbf{Q})$ is given by the two additional terms in the generalized Kirchhoff integral (6.8), plus two corresponding terms which account for the flow time-dependence and the body's deformations, namely

$$\pi(\boldsymbol{X},\mathbf{Q}) = -\frac{1}{2}\int_v V^2 \mathrm{d}v + \frac{1}{2}\int_S \phi_0' \frac{\partial \phi_0'}{\partial n}\mathrm{d}S - \int_v \frac{\partial \phi}{\partial t}\,\mathrm{d}v - \int_S \phi_d \frac{\partial \phi}{\partial n}\,\mathrm{d}S. \tag{7.4}$$

The dynamical equation corresponding to (3.13) for the translational motion is given by

$$\rho_b \frac{\mathrm{d}}{\mathrm{d}t}(v\boldsymbol{U}) + \boldsymbol{F}^{(q)} = -(\nabla\wedge\boldsymbol{K}(\phi_0'))\wedge\boldsymbol{U} + \boldsymbol{F}^{(0)}, \tag{7.5}$$

and can also be written in the familiar Euler–Lagrange form as

$$\frac{\mathrm{d}^*}{\mathrm{d}t}\left(\frac{\partial L}{\partial \boldsymbol{U}}\right) = \frac{\partial L}{\partial \boldsymbol{X}}. \tag{7.6}$$

To prove that equations (7.5) and (7.6) are indeed equivalent one needs to substitute (7.3) into (7.6) and to use (3.3), (3.5), (3.24)–(3.26), (3.31), (3.36) and (3.38). It is worth mentioning here that a similar approach is used in the Hamiltonian description of a particle motion in an electro-magnetic field (Landau & Lifshitz 1989, §16).

Applying next the Legendre transformation to the Lagrangian (7.3), leads finally to the following Hamiltonian:

$$H(\boldsymbol{X};\tilde{\boldsymbol{p}}) = \tfrac{1}{2}(\tilde{\boldsymbol{p}} - \boldsymbol{K}(\phi_0' + \phi_d')) \cdot \boldsymbol{J}^{-1}(\tilde{\boldsymbol{p}} - \boldsymbol{K}(\phi_0' + \phi_d')) + \pi(\boldsymbol{X}, \mathbf{Q} = \hat{\mathbf{1}}), \tag{7.7}$$

where

$$\tilde{\boldsymbol{p}} = \boldsymbol{p} + \boldsymbol{K}(\phi_0') = \boldsymbol{p} - \int_S \boldsymbol{\Phi}(\boldsymbol{n} \cdot V)\, \mathrm{d}S = \boldsymbol{p}_b + \boldsymbol{K}(\phi'), \tag{7.8}$$

and the body's momentum is defined as $\boldsymbol{p}_b = \rho_b v \boldsymbol{U}$. Thus, it is shown that the canonical impulse conjugated to the coordinate $\boldsymbol{X}$ for a non-rotating body is simply given by the sum of the body's momentum and the additional Kelvin impulse $\boldsymbol{K}(\phi')$ of the surrounding fluid, resulting from introducing the moving deformable body into the fluid. The corresponding Poisson brackets acquire then the canonical structure

$$[\tilde{p}_i, \tilde{p}_j] = [X_i, X_j] = 0, \qquad [X_i, \tilde{p}_j] = \delta_{ij}, \tag{7.9}$$

and the Hamiltonian equations of motion can thus be written in the traditional manner as (Olver 1986, Ch. 6)

$$\dot{X}_i = [X_i, H], \quad \dot{\tilde{p}}_i = [\tilde{p}_i, H], \tag{7.10}$$

with H given by (7.7).

It can further be proven, that for an *arbitrary* motion of a deformable body in a non-uniform non-stationary flow field, the canonical conjugated coordinates and impulses are indeed $\boldsymbol{X}, \tilde{\boldsymbol{p}}$ and $\mathbf{Q}, \tilde{\boldsymbol{l}}$ respectively. Here $\tilde{\boldsymbol{p}}$ is given by (7.8), and $\tilde{\boldsymbol{l}}$ is similarly defined as

$$\tilde{\boldsymbol{l}} = \boldsymbol{l} + \boldsymbol{P}(\phi_0') = \boldsymbol{l} - \int_S \boldsymbol{\Psi}(\boldsymbol{n} \cdot V)\, \mathrm{d}S = \boldsymbol{l}_b + \boldsymbol{P}(\phi'), \tag{7.11}$$

where $\boldsymbol{l}_b = \boldsymbol{I\Omega}$ represents the angular momentum of the rigid body and $\boldsymbol{P}(\phi')$ denotes the Kelvin-impulse couple induced in the fluid by the moving (and generally rotating) deformable body.

The corresponding Lagrangian, generalizing (7.7) for a *rotating* body, is given then by

$$L(\boldsymbol{X}, \mathbf{Q}; \boldsymbol{U}, \boldsymbol{\Omega}) = \frac{1}{2}\boldsymbol{J}\begin{vmatrix}\boldsymbol{U} \\ \boldsymbol{\Omega}\end{vmatrix} \cdot \begin{vmatrix}\boldsymbol{U} \\ \boldsymbol{\Omega}\end{vmatrix} + \boldsymbol{K}(\phi_0' + \phi_d') \cdot \boldsymbol{U} + \boldsymbol{P}(\phi_0' + \phi_d') \cdot \boldsymbol{\Omega} - \pi(\boldsymbol{X}, \mathbf{Q}), \tag{7.12}$$

which for $\boldsymbol{\Omega} = 0$ and $\mathbf{Q} = \hat{\mathbf{1}}$ clearly reduces to (7.3). Supplementing (7.6) we have now the additional Euler–Lagrange equation corresponding to the variable $\mathbf{Q}$, i.e.

$$\frac{\mathrm{d}^*}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{Q}}}\right) = \frac{\partial L}{\partial \mathbf{Q}}, \tag{7.13}$$

where only three (among the nine) equations are in fact independent. The following relationship between time-derivatives in the two coordinate systems:

$$\frac{\mathrm{d}^*}{\mathrm{d}t} = \hat{\boldsymbol{\Omega}} + \frac{\mathrm{d}}{\mathrm{d}t}, \tag{7.14}$$

should be applied in (7.13). To evaluate the terms $\partial\boldsymbol{\Omega}/\partial\dot{\mathbf{Q}}$ and $\partial\boldsymbol{\Omega}/\partial\mathbf{Q}$ in (7.13) one should use (2.23). The operator $\mathrm{d}/\mathrm{d}t$ is then calculated with the help of (2.26). By substituting the Lagrangian (7.12) in (7.6) and (7.13), one finally obtains (after some extensive calculations) the desired governing equation (3.13).

In order to illustrate the calculation procedure let us show, for example, that the term $\boldsymbol{P}(\phi_0) \cdot \boldsymbol{\Omega}$ in (7.12) gives rise to the matrix D in (3.13). Thus, starting with

$$\frac{\partial(\boldsymbol{P} \cdot \boldsymbol{\Omega})}{\partial \dot{\mathsf{Q}}} = \tfrac{1}{2}\mathsf{QP}, \text{ and } \frac{\partial(\boldsymbol{P} \cdot \boldsymbol{\Omega})}{\partial \mathsf{Q}} = \tfrac{1}{2}\dot{\mathsf{Q}}\mathsf{P}, \tag{7.15}$$

where we denote $\mathsf{P} \equiv \tau(\boldsymbol{P})$ (see (2.22)) and using next (7.15) one obtains

$$\left(\frac{\mathrm{d}^*}{\mathrm{d}t}\frac{\partial}{\partial \dot{\mathsf{Q}}} - \frac{\partial}{\partial \mathsf{Q}}\right)(\boldsymbol{P} \cdot \boldsymbol{\Omega}) = \frac{1}{2}\mathsf{Q}\frac{\mathrm{d}^*\mathsf{P}}{\mathrm{d}t} - \boldsymbol{\Omega} \cdot \frac{\partial \boldsymbol{P}}{\partial \mathsf{Q}}. \tag{7.16}$$

The term $\mathrm{d}^*\mathsf{P}/\mathrm{d}t$ is already evaluated in Appendix C as the 'unsteady moment'. Employing the continuation (2.25), one can also verify that the last term in the right-hand side of (7.16) is equal to the $\boldsymbol{\Omega}$-part of the right-hand side of (2.19) (compare with (C 16) for the 'steady' moment). This procedure leads (in a similar manner to Appendix C) to the expression for the matrix D given in (3.20) and (3.21). For reasons of brevity the details of other tedious calculations (similar to those presented in the Appendices B and C) are omitted here.

The corresponding non-local Hamiltonian for a rotating body can be finally written (using the Legendre transformation) as

$$H(\boldsymbol{X}, \mathsf{Q}; \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{l}}) = \frac{1}{2}\left|\begin{matrix}\tilde{\boldsymbol{p}} - \boldsymbol{K}(\phi_0' + \phi_d) \\ \tilde{\boldsymbol{l}} - \boldsymbol{P}(\phi_0' + \phi_d)\end{matrix}\right| \cdot \mathsf{J}^{-1}\left|\begin{matrix}\tilde{\boldsymbol{p}} - \boldsymbol{K}(\phi_0' + \phi_d) \\ \tilde{\boldsymbol{l}} - \boldsymbol{P}(\phi_0' + \phi_d)\end{matrix}\right| + \pi(\boldsymbol{X}, \mathsf{Q}). \tag{7.17}$$

It is noted here that for a rigid body embedded in a stationary flow field, the first integral $H = \text{const}$ corresponds to the energy conservation law (6.8). As a concluding remark we underline again the fundamental physical significance of the Kelvin impulse and impulse couple (resulting from the body's motion and expressed in the coordinate system moving with the body) as the natural variables in the present Hamiltonian formalism.

8. A rigid body in a stationary stream

8.1. *Rate of spreading*

One of the intrinsic characteristics of an ambient flow field is the so-called 'rate of spreading' of small particles of a given shape S. It is literally defined as the sub-domain (part of the whole flow domain) which can be reached by particles initially placed in a small neighbourhood of $\boldsymbol{X}$. Note, that *liquid* particles spread through the irrotational motion, whereas for the spreading of *rigid* particles the situation is somewhat different because a rigid body can in fact rotate. Such a motion can indeed arrest the body within a bounded domain and induce in principle (even in a quiescent flow field) a spiral-like motion with possible closed trajectories (Novikov 1981). Applications of the rate-of-spreading concept can also be found in other disciplines such as two-phase flows, chemical technology and Lagrangian chaos (e.g. Dahlen 1992).

Let us apply the first integral (6.8) in order to determine the space domain reachable by a rigid body of shape S commencing its motion from the point $X(0)$. The first integral does not include any information about the matrix W, but, nevertheless, some general useful qualitative observations can be stated, as demonstrated below.

We start with the following inequality:

$$-\int_S \phi_0' \frac{\partial \phi_0'}{\partial n} \mathrm{d}S \leqslant \lambda_1 \max_{y \in S(X)} |\boldsymbol{V}(y)|^2, \tag{8.1}$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant 0$ are the three eigenvalues of the symmetric translational added-mass tensor $\boldsymbol{M}$. The inequality (8.1) follows from the representation

$$-\int_S \phi_0' \frac{\partial \phi_0}{\partial n} \mathrm{d}S = \int_S \int_S G^{(out)}(\boldsymbol{x},\boldsymbol{y})\,(\boldsymbol{V}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}))\,(\boldsymbol{V}(\boldsymbol{y}) \cdot \boldsymbol{n}(\boldsymbol{y}))\,\mathrm{d}S(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y}), \tag{8.2}$$

and the inequality

$$\int_S \int_S \boldsymbol{K}(\boldsymbol{x},\boldsymbol{y})\mathbf{f}(\boldsymbol{x}) \cdot \mathbf{f}(\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y}) \leqslant |\boldsymbol{K}| \max_{\boldsymbol{x}\in S} |\mathbf{f}(\boldsymbol{x})|^2, \tag{8.3}$$

which can be defined for any compact self-adjoint operator $\boldsymbol{K}$ with a norm $|\boldsymbol{K}|$ acting in the space of the quadratically summarizable functions $f(\boldsymbol{x})$ on S. Since

$$\int_{v(t)} V^2 \mathrm{d}v \leqslant v \left(\max_{\boldsymbol{x}\in S(t)} V^2 \right), \tag{8.4}$$

one finally obtains

$$\pi(t) \geqslant -\tfrac{1}{2}(v + \lambda_1) \left(\max_{\boldsymbol{x}\in S(t)} V^2 \right), \tag{8.5}$$

where the potential $\pi(t)$ is defined for a rigid body embedded in a stationary flow field (see (7.4)) as

$$\pi(t) \equiv -\frac{1}{2} \int_v V^2 \mathrm{d}v + \frac{1}{2} \int_S \phi_0' \frac{\partial \phi_0}{\partial n} \mathrm{d}S. \tag{8.6}$$

Here $S(t)$ actually means $S(\boldsymbol{X}(t))$, i.e. the centroid of the body is located at the point $\boldsymbol{X}(t)$.

Next, we rewrite the first integral (6.8) in the following form:

$$E_b(t) + \pi(t) = E_b(0) + \pi(0), \tag{8.7}$$

where the effective kinetic energy of the body (expressed in terms of its added-mass tensor) is given by

$$E_b(t) \equiv \tfrac{1}{2}(\rho_b v \hat{\mathbf{1}} + \boldsymbol{M})\boldsymbol{U} \cdot \boldsymbol{U} + \tfrac{1}{2}(\boldsymbol{I} + \boldsymbol{R})\boldsymbol{\Omega} \cdot \boldsymbol{\Omega} + \boldsymbol{Z}\boldsymbol{U} \cdot \boldsymbol{\Omega}. \tag{8.8}$$

Since $E_b \geqslant 0$, it follows from (8.7) that

$$\pi(t) \leqslant E_b(0) + \pi(0), \tag{8.9}$$

or, by using (8.5)

$$\tfrac{1}{2}(v + \lambda_1) \left(\max_{\boldsymbol{x}\in S(t)} V^2 \right) \geqslant E_b(t) + \pi(0). \tag{8.10}$$

Hence, for a body which is *initially at rest* (i.e. $E_b(0) = 0$), the possible reachable domain of motion is bounded by

$$\left(\max_{\boldsymbol{x}\in S(t)} V^2(\boldsymbol{X} + \boldsymbol{x}) \right) \geqslant -\frac{2\pi(0)}{v + \lambda_1}. \tag{8.11}$$

It also follows from (8.11) that if $|\boldsymbol{V}|(\boldsymbol{X}) \to 0$ when $|\boldsymbol{X}| \to \infty$, then the motion of an initially stationary body is always bounded. Indeed, the left-hand side of (8.11) tends to zero at infinity whereas the right-hand side always remains positive. This clearly contradicts the inequality (8.11). The inequality (8.11) determines the space sub-domain which cannot be reached by an initially stationary rigid particle which is released at $\boldsymbol{X}(0)$. Thus, the sub-domains for which (8.11) is not valid are unreachable by the particles because the rigid body does not have enough kinetic energy to get there. One can deduce therefore from (8.11) that the larger is the first eigenvalue λ_1 (for a given v), the higher is the spreading rate of rigid particles. Physically speaking, an elongated body is more likely to align itself along the direction of the streamlines of the ambient flow. It also follows from (8.11) that particles passing near stagnation zones generally experience a larger spreading rate, such as, for example, the motion in the velocity field induced by a Koda vortex (see Dahlen 1992).

8.2. *Bounds on body's velocities*

Invoking the first integral (6.8), one can show interestingly enough that the velocities of the rigid body are all bounded and are correlated with the velocity of the ambient flow field, evaluated on the body's surface. Thus, using (8.7) and (8.5) one obtains

$$E_b(t) = E_b(0) + \pi(0) - \pi(t) \leqslant \tfrac{1}{2}(v + \lambda_1)\left(\max_{x \in S(t)} V^2\right) + c_0, \tag{8.12}$$

where the initial parameter c_0 is defined as

$$c_0 = E_b(0) + \pi(0). \tag{8.13}$$

Correspondingly, it follows from (8.12) and the definition (8.8) that the modulus of the body's velocity is always bounded from above by

$$|\boldsymbol{U}|(\boldsymbol{X}(t)) \leqslant \left(\frac{v + \lambda_1}{v\rho_b + \lambda_3}\right)^{1/2} \left(\max_{x \in S(t)} |\boldsymbol{V}|(\boldsymbol{X}(t) + \boldsymbol{x})\right) + |\boldsymbol{U}|(0). \tag{8.14}$$

A similar upper bound can be also found for the angular velocity, i.e.

$$|\boldsymbol{\Omega}|(\boldsymbol{X}(t)) \leqslant \left(\frac{v + \lambda_1}{\mu_3}\right)^{1/2} \left(\max_{x \in S(t)} |\boldsymbol{V}|(\boldsymbol{X}(t) + \boldsymbol{x})\right) + |\boldsymbol{\Omega}|(0), \tag{8.15}$$

where $\mu_1 \geqslant \mu_2 \geqslant \mu_3 \geqslant 0$ are the three positive eigenvalues of the tensor $\boldsymbol{I} + \boldsymbol{R}$.

For the purpose of illustration of the general bounds thus found, we consider an initially stationary spherical bubble, located at a stagnation point (where $c_0 \ll \int_{v(t)} V^2 \mathrm{d}v$), for which $\rho_b = 0$, $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2}v$ and $E_b = 0$. Equation (8.14) implies then that

$$\sqrt{2}\left(\frac{\int_{v(t)} V^2 \mathrm{d}v}{v}\right)^{1/2} \leqslant |\boldsymbol{U}|(\boldsymbol{X}(t)) \leqslant \sqrt{3} \max_{x \in S(t)} |\boldsymbol{V}|, \tag{8.16}$$

where the lower bound for a non-rotating body follows from (8.7) and the inequality

$$-\frac{1}{2}\int_v V^2 \, \mathrm{d}v \geqslant \pi(t). \tag{8.17}$$

Thus, following (8.16) the motion of a rigid spherical particle physically resembles that of a liquid particle.

9. Ellipsoidal family

9.1. *Ellipsoids*

The ellipsoidal family is the simplest class of three-dimensional smooth shapes for which the foregoing analysis can be further simplified. On the other hand, it is well known that small bubbles cease to be spherical beyond a certain size and resemble instead oblate spheroidal shapes. The main property of the motion of such forms in a quiescent fluid is the tendency towards chaotization of the motion for nearly all shapes (see Aref & Jones 1993), except for some degenerate cases (such as spheres or Clebsh forms (Kozlov & Onichenko 1982)). The fact that the translation of the body cannot in principal be separated from the rotational motion when moving in a heavy fluid (in contrast with rigid body dynamics in vacuum) manifests itself in the above-mentioned possible chaotization of the rotational and translational motions and in the resulting permanent redistribution of kinetic energy between all degrees of freedom.

Let us consider, for example, an ellipsoidal quadratic shape. Substituting (5.3) in (3.25) and using the Gauss theorem yields

$$\boldsymbol{K}(\acute{\phi}_0) = -\frac{1}{v}\boldsymbol{M}\int_v \boldsymbol{V}\,\mathrm{d}v. \tag{9.1}$$

Hence, using (3.24) one derives

$$\boldsymbol{A}\boldsymbol{U} = -(\nabla_X \wedge \boldsymbol{K}(\acute{\phi}_0)) \wedge \boldsymbol{U} = -\frac{1}{v}(\nabla_X \wedge \boldsymbol{M}\boldsymbol{K}(\phi)) \wedge \boldsymbol{U} = \frac{1}{v}\left(\boldsymbol{M}\int_v \boldsymbol{E}\,\mathrm{d}v - \int_v \boldsymbol{E}\,\mathrm{d}v\,\boldsymbol{M}\right)\boldsymbol{U}, \tag{9.2}$$

which suggests that the matrix $\boldsymbol{A}$ for an ellipsoidal shape can be simply expressed in terms of the Kelvin impulse of the ambient flow field. Clearly, if the Kelvin impulse $\boldsymbol{K}(\phi)$ is an eigenvector of $\boldsymbol{M}$ (say, the ambient stream is aligned along the direction of one of the added-mass eigenvectors), then $\boldsymbol{A}$ identically vanishes.

9.2. *The governing equation for a sphere*

For a sphere one obtains $\boldsymbol{M} = \frac{1}{2}v\hat{\mathbf{1}}$, $\boldsymbol{\Psi} = 0$ and by virtue of the last identity in the right-hand side of (9.2), it follows that

$$(\boldsymbol{A})_{sph} = 0. \tag{9.3}$$

Similarly, one obtains from (3.20) that $\boldsymbol{D}$ also vanishes and the consequent use of the Gauss theorem and the mean theorem of harmonic functions for (3.17) lead to $\boldsymbol{B} = \boldsymbol{C} = 0$. Thus, $\boldsymbol{W}_{sph} = 0$, since sphere's rotation is immaterial in a perfect flow field.

Using, next the Weiss theorem (Milne-Thomson 1968) for $\acute{\phi}_0$, i.e.

$$\acute{\phi}_0(\boldsymbol{x})\Big|_S = \phi\Big|_S - \int_0^1 \phi(\sigma\boldsymbol{x})\,\mathrm{d}\sigma\Big|_S, \tag{9.4}$$

in conjunction with (3.31), (3.36) and (5.8), results in the following expression for the force acting on a stationary sphere with a time-dependent radius $r = a(t)$:

$$\boldsymbol{F}^{(0)} = \frac{3}{2}\int_v \frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t}\mathrm{d}v + \int_v \boldsymbol{E}(\boldsymbol{x})\left(\int_0^1 \sigma(\boldsymbol{V}(\boldsymbol{x}) - \boldsymbol{V}(\sigma\boldsymbol{x}))\,\mathrm{d}\sigma\right)\mathrm{d}v + \frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{V}. \tag{9.5}$$

The corresponding generalized Kirchhoff equation is finally obtained in the form

$$v(\rho_b + \tfrac{1}{2})\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} = \frac{3}{2}\int_v \frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t}\mathrm{d}v + \int_v \boldsymbol{E}(\boldsymbol{x})\left(\int_0^1 \sigma\,(\boldsymbol{V}(\boldsymbol{x}) - \boldsymbol{V}(\sigma\boldsymbol{x}))\mathrm{d}\sigma\right)\mathrm{d}v + \frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{V}, \tag{9.6}$$

where all variables are now expressed in the *laboratory* coordinate system, because the rotation of the sphere does not enter into the formulation (see also Biesheuvel 1985).

For a translating sphere ($\boldsymbol{\Omega} = 0$), (9.6) can be written in the form of a Newtonian equation of motion for a rigid particle placed in a time-dependent effective potential field $\pi_{sph}(\boldsymbol{X},t)$, which depends quadratically on the ambient flow field $\boldsymbol{V}$ (compare with (7.5)), i.e.

$$v(\rho_b + \tfrac{1}{2})\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{X} = -\nabla\pi_{sph}(\boldsymbol{X},t), \tag{9.7}$$

where the potential π_{sph} is given by

$$\pi_{sph}(\boldsymbol{X},t) \equiv -\frac{3}{4}\int_v \boldsymbol{V}^2 \mathrm{d}v - \frac{1}{2}\int_0^1 \sigma\left(\int_v \boldsymbol{V}(\boldsymbol{x})\cdot(\boldsymbol{V}(\boldsymbol{x}) - \boldsymbol{V}(\sigma\boldsymbol{x}))\,\mathrm{d}v\right)\mathrm{d}\sigma - \frac{3}{2}v\frac{\partial\phi}{\partial t} - \frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}t}\phi. \tag{9.8}$$

For a rigid spherical shape embedded in a *weakly* non-uniform flow field, the last integral in the right-hand side of (9.6) is clearly of $O(\epsilon^2)$. This can be also verified by pulling $\boldsymbol{E}$ in front of the integrals and using the mean theorem for the harmonic function $\boldsymbol{V}$. Thus, for a rigid sphere placed in a weakly non-uniform stream, one obtains from (9.6)–(9.8)

$$|\boldsymbol{U}| = \left(\frac{3}{1+2\rho_b}\right)^{1/2}\left(\boldsymbol{V}^2(\boldsymbol{X}(t)) - \boldsymbol{V}^2(\boldsymbol{X}(0)) + \frac{1+2\rho_b}{3}\boldsymbol{U}^2(0)\right)^{1/2} + O(\epsilon^2), \tag{9.9}$$

which, for a massless bubble($\rho_b = 0$) coincides exactly with Auton, Hunt & Prud'homme (1988, eq. (4.1)). For a neutrally buoyant ($\rho_b = 1$) sphere letting $\boldsymbol{U}(0) = \boldsymbol{V}(0)$ implies that $\boldsymbol{U}(t) = \boldsymbol{V}(t)$. The equation of motion (9.6) reduces in this case to the Euler equation for a liquid particle.

9.3. *Ambient flow fields with a symmetry*

The proposed Hamiltonian formalism of the body's motion also suggests that if a certain symmetry of the body's surface coincides with that of the ambient flow field, then an additional integral of motion must also arise. Thus, if, for example, the imposed flow field $\boldsymbol{V}$ has a central symmetry, then the effective potential π_{sph} for a sphere also inherits a central symmetry. In this case the problem appears to be completely integrable, similar to the corresponding classical problem in rigid-body dynamics.

Consider, for example, an exterior source (sink) with an output $s(t)$ lying in the proximity of a pulsating sphere at the point $\acute{\boldsymbol{X}}$. One can show then that

$$\pi_{sph} \sim -\frac{s^2}{|\acute{\boldsymbol{X}} - \boldsymbol{X}|^4} - \frac{f(t)}{|\acute{\boldsymbol{X}} - \boldsymbol{X}|}, \tag{9.10}$$

where

$$f \equiv -\frac{1}{2}\left(3\frac{\mathrm{d}s}{\mathrm{d}t}v + s\frac{\mathrm{d}v}{\mathrm{d}t}\right), \tag{9.11}$$

which represents an attraction force near the source (sink). Nevertheless, far from the body, the second term in (9.10) starts to dominate and the total force can be therefore of repulsion or attraction, depended on the sign of f. It is also interesting to note that after time averaging, the term $\langle f\rangle = \langle v\mathrm{d}s/\mathrm{d}t\rangle$, is generally non-zero and gives rise to the common Bjerkness force in bubble dynamics.

Moreover, the symmetry of π_{sph} can be of a higher degree than the symmetry of the ambient flow field. Thus, the potential of a simple gravity wave (in a coordinate system moving with the wave) is given by $\phi \sim \mathrm{e}^{-(kz)}\cos(k_1x + k_2y)$, where $k^2 = k_1^2 + k_2^2$. The direct calculations using (9.8) show no dependence of π_{sph} on the x- and y-coordinates, which implies that the sphere's impulse in the x, y-directions is conserved.

One can also consider the common case of an axisymmetric body, initially aligned along the axis of symmetry of an axisymmetric external flow. It can then be readily shown that the dynamical system is fully integrable, provided the initial velocities $\boldsymbol{U}(0)$ and $\boldsymbol{\Omega}(0)$ of the body are collinear with the axis of symmetry of the imposed stream. Indeed, the resulting motion of the body is essentially one-dimensional and therefore the first integral of motion (6.8) is precisely the one needed to perform the full integration.

As a final remark we note that by expanding $\boldsymbol{V}$ in (9.6) in a Taylor series about the sphere's centre, one recovers the analytic expression recently obtained by Miloh (1994) for the axial force experienced by a rigid sphere placed in a non-uniform axisymmetric flow field.

10. Summary and conclusions

We have derived here the corresponding system of six nonlinear ordinary differential equations of the second order which govern the motion of a *deformable* body, embedded in a perfect *non-uniform non-stationary* ambient flow field. The newly obtained dynamical system of equations generalizes the well-known Kirchhoff equations corresponding to the motion of a *rigid* body placed in a *quiescent* flow field. It is proven that this general system exhibits a very special antisymmetric (gyroscopic) property.

A first integral of motion is shown to always exist for a *rigid* body moving in a *stationary* flow field. This first integral has the physical meaning of an energy-conservation principle and hence reflects the time-symmetry property of the combined system (without memory) comprising 'rigid body plus stationary potential stream'. If the surrounding fluid is otherwise at rest (i.e. $\boldsymbol{V} = 0$), the first integral reduces to the conventional Hamiltonian (Novikov 1981).

It is further demonstrated that a general motion of a deformable body (i.e. one which combines translation, rotation and deformation) embedded in an arbitrary ambient flow field is always Hamiltonian. The explicit form of the corresponding *non-local* Hamiltonian is also determined. The canonical generalized impulses are conjugated to the coordinates of the body's centroid $\boldsymbol{X}$ and the matrix $\boldsymbol{Q}$ (connecting the laboratory and the body-attached coordinate systems). The canonical generalized impulses are expressed as a sum of the body's linear/angular momentum and the Kelvin impulse/impulse couple of the ambient flow resulting from the body's rigid and deformation motion.

For the case of a *rigid* body placed in a *stationary* stream, the corresponding Hamiltonian coincides with the above-mentioned first integral of motion, written in terms of the generalized coordinates $\boldsymbol{X}, \boldsymbol{Q}$ and the generalized impulses. Note, also that the motion of a non-rotating deformable body is mathematically equivalent to the motion of a non-isotropic particle embedded in an effective non-stationary magnetic and electrical field. Thus, for the particular common case of a pulsating spherical shape, the motion in a non-uniform stream is similar to the motion of a particle in an effective potential force field which quadratically depends on the ambient velocity field.

As a direct consequence of the existence of the first integral, it is proven that the motion of an initially stationary rigid body is always bounded, regardless of the shape of the ambient flow pathlines. It is further noted that the spreading rate of rigid particles is larger for particles which are injected near stagnation points. It is also remarked that an elongated shape is generally the preferable shape, when trying to maximize the larger spreading rate.

In addition, it is proven that the presence of an ambient non-stationary flow field tends to considerably enhance the effect of a persistent self-propulsion. Following Miloh & Galper (1993), a deformable (non-isochoric) spherical shape is shown to be the 'worst' self-propulsor. However, it is demonstrated that even a sphere undergoing only simple symmetric volume deformations can indeed develop a persistent velocity of self-propulsion, as a result of nonlinear interactions with the surrounding non-homogeneous *stationary* flow field. Here the self-propulsion phenomenon appears to be a manifestation of *parametric* resonant interactions between the body's deformation and the flow non-uniformity. Thus, as an example, the 'dancing-bubble' effect analysed by Benjamin & Ellis (1990) for a *single* bubble, is much more pronounced in a bubble cloud. This is not only as a result of the symmetry-breaking phenomena (Miloh & Galper 1993), but also due to the additional parametric resonant mechanism between a single bubble and the effective flow non-uniformity in a cloud.

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Appendix A. Derivation of the classical Kirchhoff equations for a deformable body

The force $-\boldsymbol{F}^{(q)}$ acting on a deformable body in a quiescent fluid is given by the first term in the right-hand side of (2.11), as

$$-\boldsymbol{F}^{(q)} = \frac{\mathrm{d}^*}{\mathrm{d}t}\int_S (\boldsymbol{U}\cdot\boldsymbol{\Phi} + \boldsymbol{\Psi}\cdot\boldsymbol{\Omega} + \phi_d)\cdot\boldsymbol{n}\,\mathrm{d}S. \tag{A 1}$$

In accordance with (3.7) and (3.9) one obtains

$$\int_S (\boldsymbol{U}\cdot\boldsymbol{\Phi} + \boldsymbol{\Psi}\cdot\boldsymbol{\Omega} + \phi_d)\cdot\boldsymbol{n}\,\mathrm{d}S = -(\boldsymbol{M}\boldsymbol{U} + \boldsymbol{Z}\boldsymbol{\Omega} + \boldsymbol{K}(\phi_d)). \tag{A 2}$$

Using finally (7.14) one derives the Kirchhoff term $\boldsymbol{F}^{(q)}$ for a deformable body given by (3.3) and (3.5). Correspondingly $-\boldsymbol{M}^{(q)}$ for a quiescent fluid is expressed by the first two terms of (2.12), i.e.

$$-\boldsymbol{M}^{(q)} = \frac{\mathrm{d}^*}{\mathrm{d}t}\int_S (\boldsymbol{U}\cdot\boldsymbol{\Phi}+\boldsymbol{\Omega}\cdot\boldsymbol{\Psi}+\phi_d)\,\boldsymbol{x}\wedge\boldsymbol{n}\,\mathrm{d}S + \boldsymbol{U}\wedge\int_S (\boldsymbol{U}\cdot\boldsymbol{\Phi}+\boldsymbol{\Omega}\cdot\boldsymbol{\Psi}+\phi_d)\,\boldsymbol{n}\,\mathrm{d}S. \tag{A 3}$$

Enforcing then (3.7) and (3.9) yields

$$\frac{\mathrm{d}^*}{\mathrm{d}t}\int_S (\boldsymbol{U}\cdot\boldsymbol{\Phi} + \boldsymbol{\Omega}\cdot\boldsymbol{\Psi} + \phi_d)\,\boldsymbol{x}\wedge\boldsymbol{n}\,\mathrm{d}S = -\frac{\mathrm{d}^*}{\mathrm{d}t}\left(\boldsymbol{Z}^T\boldsymbol{U} + \boldsymbol{R}\boldsymbol{\Omega} + \boldsymbol{P}(\phi_d)\right). \tag{A 4}$$

Using finally (A 2) and (A 4) one obtains the desired expressions (3.4) and (3.6) for the Kirchhoff moment acting on a deformable body.

Appendix B. Evaluation of expressions (3.17) for $\boldsymbol{A}$ and (3.16) for $\boldsymbol{B}$

B.1. *The unsteady force*

In order to evaluate (3.16) and (3.17) let us first calculate the unsteady part of the force, given by the first term in the right-hand side of (2.11) (see Appendix A) as

$$\boldsymbol{F}_{un} = -\boldsymbol{F}^{(q)} + \frac{\mathrm{d}^*}{\mathrm{d}t}\int_S \phi_0'\boldsymbol{n}\,\mathrm{d}S + \frac{\mathrm{d}^*}{\mathrm{d}t}\int_S \phi\boldsymbol{n}\,\mathrm{d}S, \tag{B 1}$$

where we recall that $V \equiv \nabla_X \phi(X)$ and that $\mathrm{d}^*/\mathrm{d}t$ represents the time derivative in the laboratory coordinate system.

It is further noted that

$$\frac{\mathrm{d}^*}{\mathrm{d}t}\int_S \phi_0 \boldsymbol{n} \,\mathrm{d}S = -\frac{\mathrm{d}^*}{\mathrm{d}t}\int_S \boldsymbol{m}\boldsymbol{V}\,\mathrm{d}S,$$

where $\boldsymbol{m}$ is given by (3.10). Hence, because of (2.8), and using (7.14) one obtains

$$\frac{\mathrm{d}^*}{\mathrm{d}t}\int_S \boldsymbol{m}\boldsymbol{V}\,\mathrm{d}S = \boldsymbol{\Omega}\wedge\int_S \boldsymbol{m}\boldsymbol{V}\,\mathrm{d}S + \frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{m}\boldsymbol{V}\,\mathrm{d}S. \tag{B 2}$$

The vector field $\boldsymbol{V}(\boldsymbol{x})$ in (B 2) is determined in accordance with (2.25) and (2.26), from where one can immediately deduce using (2.23) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{m}\boldsymbol{V}\,\mathrm{d}S &= -\int_S \boldsymbol{m}(\boldsymbol{\Omega}\wedge\boldsymbol{V})\,\mathrm{d}S + \int_S \boldsymbol{m}\mathbf{Q}^T\boldsymbol{E}^*\left(\boldsymbol{U}^* + \mathbf{Q}(\boldsymbol{\Omega}\wedge\boldsymbol{x})\right)\,\mathrm{d}S \\ &\quad + \int_S \boldsymbol{m}\frac{\partial \boldsymbol{V}}{\partial t}\,\mathrm{d}S + \frac{\delta}{\delta t}\int_S \boldsymbol{m}\boldsymbol{V}\,\mathrm{d}S \\ &= -\int_S \boldsymbol{m}(\boldsymbol{\Omega}\wedge\boldsymbol{V})\,\mathrm{d}S + \int_S \boldsymbol{m}\boldsymbol{E}(\boldsymbol{\Omega}\wedge\boldsymbol{x})\,\mathrm{d}S + \int_S \boldsymbol{m}\boldsymbol{E}\boldsymbol{U}\,\mathrm{d}S \\ &\quad + \int_S \boldsymbol{m}\frac{\partial \boldsymbol{V}}{\partial t}\,\mathrm{d}S + \frac{\delta}{\delta t}\int_S \boldsymbol{m}\boldsymbol{V}\,\mathrm{d}S.\end{aligned} \tag{B 3}$$

The rate-of-strain tensor $\boldsymbol{E}^*$ is defined in the laboratory coordinate system as

$$\boldsymbol{E}^* \equiv \nabla_{X^*}\cdot\boldsymbol{V}^*, \qquad \boldsymbol{E} = \mathbf{Q}^T\boldsymbol{E}^*\mathbf{Q}, \tag{B 4}$$

and the 'deformation' time-derivative operator in (B 3), (i.e. time-dependence only due to pure deformations) is denoted here by $\delta/\delta t$. Gathering then (B 2) and (B 3) one obtains

$$\begin{aligned}\frac{\mathrm{d}^*}{\mathrm{d}t}\int_S \boldsymbol{m}\boldsymbol{V}\,\mathrm{d}S &= \int_S (\boldsymbol{\Omega}\wedge\boldsymbol{m}\boldsymbol{V} - \boldsymbol{m}(\boldsymbol{\Omega}\wedge\boldsymbol{V}))\,\mathrm{d}S + \int_S \boldsymbol{m}\boldsymbol{E}(\boldsymbol{\Omega}\wedge\boldsymbol{x} + \boldsymbol{U})\,\mathrm{d}S \\ &\quad + \int_S \boldsymbol{m}\frac{\partial \boldsymbol{V}}{\partial t}\,\mathrm{d}S + \frac{\delta}{\delta t}\int_S \boldsymbol{m}\boldsymbol{V}\,\mathrm{d}S.\end{aligned} \tag{B 5}$$

In a similar manner, it can be shown that

$$\frac{\mathrm{d}^*}{\mathrm{d}t}\int_S \boldsymbol{n}\phi(\boldsymbol{x})\,\mathrm{d}S = \boldsymbol{\Omega}\wedge\int_S \boldsymbol{n}\phi\,\mathrm{d}S + \frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{n}\phi\,\mathrm{d}S, \tag{B 6}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(\boldsymbol{x}) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\phi(\boldsymbol{X} + \mathbf{Q}\boldsymbol{x}) = \boldsymbol{\Omega}\cdot(\boldsymbol{x}\wedge\boldsymbol{V}) + \boldsymbol{V}\cdot\boldsymbol{U} + \frac{\partial\phi}{\partial t}. \tag{B 7}$$

Thus,

$$\begin{aligned}\frac{\mathrm{d}^*}{\mathrm{d}t}\int_S \boldsymbol{n}\phi(\boldsymbol{x})\,\mathrm{d}S &= \boldsymbol{\Omega}\wedge\int_S \boldsymbol{n}\phi(\boldsymbol{x})\,\mathrm{d}S - \int_S \boldsymbol{n}(\boldsymbol{x}\cdot(\boldsymbol{\Omega}\wedge\boldsymbol{V}))\,\mathrm{d}S + \int_S \boldsymbol{n}(\boldsymbol{V}\cdot\boldsymbol{U})\,\mathrm{d}S \\ &\quad + \int_S \boldsymbol{n}\frac{\partial\phi}{\partial t}\,\mathrm{d}S + \frac{\delta}{\delta t}\int_S \boldsymbol{n}\phi\,\mathrm{d}S.\end{aligned} \tag{B 8}$$

Taking then into account the fact that $\nabla_x\phi(\mathbf{Q}\boldsymbol{x}) = \mathbf{Q}^T\nabla_{X^*}\phi(\boldsymbol{x}^*) = \boldsymbol{V}(\boldsymbol{x})$, and using the

Gauss theorem, leads to

$$\frac{\mathrm{d}^*}{\mathrm{d}t}\int_S \boldsymbol{n}\phi(\boldsymbol{x})\,\mathrm{d}S = \int_S \boldsymbol{n}(\boldsymbol{V}\cdot\boldsymbol{U})\,\mathrm{d}S + \int_v \mathbf{E}(\boldsymbol{\Omega}\wedge\boldsymbol{x})\,\mathrm{d}v + \int_v \frac{\partial \boldsymbol{V}}{\partial t}\mathrm{d}v + \frac{\delta}{\delta t}\int_v \boldsymbol{V}\mathrm{d}v. \tag{B 9}$$

Finally, by substituting (B 9) and (B 3) into (B 1), one obtains

$$\begin{aligned}\boldsymbol{F}_{un} = -\boldsymbol{F}^{(q)} &+ \int_S \boldsymbol{n}(\boldsymbol{V}\cdot\boldsymbol{U})\,\mathrm{d}S - \int_S \boldsymbol{m}\mathbf{E}\boldsymbol{U}\,\mathrm{d}S \\ &+ \int_v \mathbf{E}(\boldsymbol{\Omega}\wedge\boldsymbol{x})\mathrm{d}v + \int_S (\boldsymbol{m}(\boldsymbol{\Omega}\wedge\boldsymbol{V}) - \boldsymbol{\Omega}\wedge\boldsymbol{m}\boldsymbol{V})\,\mathrm{d}S - \int_S \boldsymbol{m}\mathbf{E}(\boldsymbol{\Omega}\wedge\boldsymbol{x})\,\mathrm{d}S \\ &+ \int_v \frac{\partial\boldsymbol{V}}{\partial t}\mathrm{d}v - \int_S \boldsymbol{m}\frac{\partial\boldsymbol{V}}{\partial t}\,\mathrm{d}S + \frac{\delta}{\delta t}\left(\int_v \boldsymbol{V}\mathrm{d}v - \int_S \boldsymbol{m}\boldsymbol{V}\,\mathrm{d}S\right),\end{aligned} \tag{B 10}$$

which should be added to $\boldsymbol{F}_{st}$ in order to obtain the expressions sought for $\mathbf{A}\boldsymbol{U}$ and $\mathbf{B}\boldsymbol{\Omega}$.

B.2. *The steady force*

Using (2.18) for the steady part of the force and subtracting from (2.18) terms proportional to $\boldsymbol{U}$ (denoted here by $\mathbf{A}_{st}\boldsymbol{U}$), gives

$$\mathbf{A}_{st}\boldsymbol{U} = \int_S \left(\mathbf{E}\boldsymbol{m}^T\boldsymbol{U} - (\boldsymbol{U}\cdot\boldsymbol{n})\boldsymbol{V}\right)\,\mathrm{d}S. \tag{B 11}$$

Combining (B 11) for the terms containing $\boldsymbol{U}$ in the steady part of the force and adding (B 10) for the $\boldsymbol{U}$-terms in the unsteady part, renders

$$\mathbf{A} = \int_S (\mathbf{E}\boldsymbol{m}^T - \boldsymbol{m}\mathbf{E})\,\mathrm{d}S, \tag{B 12}$$

which is exactly the desired expression (3.16). Correspondingly, the contribution from the $\boldsymbol{\Omega}$-terms in (2.18) (denoted here by $\mathbf{B}_{st}\boldsymbol{\Omega}$), is given by

$$\mathbf{B}_{st}\boldsymbol{\Omega} = \int_S (\boldsymbol{\Omega}\cdot\boldsymbol{\Psi})\frac{\partial\boldsymbol{V}}{\partial n}\,\mathrm{d}S + \int_S \boldsymbol{V}\,(\boldsymbol{n}\cdot(\boldsymbol{x}\wedge\boldsymbol{\Omega}))\,\mathrm{d}S. \tag{B 13}$$

The last term in (B 13) can be also written by using the Gauss theorem as

$$\int_S \boldsymbol{V}\,(\boldsymbol{n}\cdot(\boldsymbol{x}\wedge\boldsymbol{\Omega}))\,\mathrm{d}S = \int_v \mathbf{E}(\boldsymbol{x}\wedge\boldsymbol{\Omega})\,\mathrm{d}v \tag{B 14}$$

which is shown to cancel a similar term in $\boldsymbol{F}_{un}$ (see (B 10)). Thus, finally one obtains

$$\mathbf{B}\boldsymbol{\Omega} = \int_S (\boldsymbol{m}(\boldsymbol{\Omega}\wedge\boldsymbol{V}) - \boldsymbol{\Omega}\wedge\boldsymbol{m}\boldsymbol{V})\,\mathrm{d}S - \int_S \boldsymbol{m}\mathbf{E}(\boldsymbol{\Omega}\wedge\boldsymbol{x})\,\mathrm{d}S + \int_S \mathbf{E}\boldsymbol{z}^T\boldsymbol{\Omega}\,\mathrm{d}S, \tag{B 15}$$

which is identical with (3.17).

Appendix C. Evaluation of expressions (3.18) for C and (3.20) for D

According to (2.12), the total moment can be separated into a 'steady' and an 'unsteady' components. The steady part $\boldsymbol{M}_{st}$, is defined by (2.19), whereas the unsteady component $\boldsymbol{M}_{un}$, is represented by the first two terms of (2.12), i.e.

$$\boldsymbol{M}_{un}(\phi) \equiv \frac{\mathrm{d}^*}{\mathrm{d}t}\int_S (\phi + \acute{\phi})\boldsymbol{x}\wedge\boldsymbol{n}\,\mathrm{d}S + \boldsymbol{U}\wedge\int_S (\phi+\acute{\phi})\boldsymbol{n}\,\mathrm{d}S, \tag{C 1}$$

where $\acute{\phi}$ is given by (2.7). The unsteady part of the moment can be evaluated in a similar manner to the unsteady part of the force as described in Appendix B, §B.1.

C.1. *The unsteady moment*

Let us use (2.7), (2.8) and Appendix A in order to rewrite (C 1) as

$$\frac{\mathrm{d}^*}{\mathrm{d}t}\int_S \acute{\phi}\boldsymbol{x}\wedge\boldsymbol{n}\,\mathrm{d}S+\frac{\mathrm{d}^*}{\mathrm{d}t}\int_S \phi\boldsymbol{x}\wedge\boldsymbol{n}\,\mathrm{d}S+\boldsymbol{U}\wedge\int_S(\phi+\acute{\phi})\boldsymbol{n}\,\mathrm{d}S$$

$$=-\boldsymbol{M}^{(q)}-\frac{\mathrm{d}^*}{\mathrm{d}t}\int_S \boldsymbol{\Psi}(\boldsymbol{n}\cdot\boldsymbol{V})\,\mathrm{d}S+\frac{\mathrm{d}^*}{\mathrm{d}t}\int_S \phi(\boldsymbol{x}\wedge\boldsymbol{n})\,\mathrm{d}S$$
$$+\boldsymbol{U}\wedge\int_S \phi\boldsymbol{n}\,\mathrm{d}S-\boldsymbol{U}\wedge\int_S \boldsymbol{\Phi}(\boldsymbol{n}\cdot\boldsymbol{V})\,\mathrm{d}S. \tag{C 2}$$

Next, notice that

$$\frac{\mathrm{d}^*}{\mathrm{d}t}\phi(\boldsymbol{x})=\frac{\mathrm{d}^*}{\mathrm{d}t}\phi(\boldsymbol{Q}\boldsymbol{x}+\boldsymbol{X}^*(t))=\nabla_{X^*}\phi(\boldsymbol{x}^*)\boldsymbol{Q}(\boldsymbol{\Omega}\wedge\boldsymbol{x})+\boldsymbol{V}(\boldsymbol{x})\cdot\boldsymbol{U}+\frac{\partial\phi}{\partial t}, \tag{C 3}$$

from which we deduce that

$$\frac{\mathrm{d}^*}{\mathrm{d}t}\int_S \phi(\boldsymbol{x}\wedge\boldsymbol{n})\,\mathrm{d}S=\boldsymbol{\Omega}\wedge\int_S \phi(\boldsymbol{x}\wedge\boldsymbol{n})\,\mathrm{d}S+\int_S(\boldsymbol{x}\wedge\boldsymbol{n})\boldsymbol{\Omega}\cdot(\boldsymbol{x}\wedge\boldsymbol{V})\,\mathrm{d}S+\int_S(\boldsymbol{x}\wedge\boldsymbol{n})(\boldsymbol{V}\cdot\boldsymbol{U})\,\mathrm{d}S$$
$$+\int_S\frac{\partial\phi}{\partial t}(\boldsymbol{x}\wedge\boldsymbol{n})\,\mathrm{d}S+\frac{\delta}{\delta t}\int_S\phi(\boldsymbol{x}\wedge\boldsymbol{n})\mathrm{d}S. \tag{C 4}$$

Furthermore, by using the Gauss theorem, the first two integrals in the right-hand side of (C 4) can be written as

$$\boldsymbol{\Omega}\wedge\int_S\phi\,(\boldsymbol{x}\wedge\boldsymbol{n})\,\mathrm{d}S=\boldsymbol{\Omega}\wedge\int_v\boldsymbol{x}\wedge\boldsymbol{V}\mathrm{d}v,$$

and

$$\int_S(\boldsymbol{x}\wedge\boldsymbol{n}))\boldsymbol{\Omega}\cdot(\boldsymbol{x}\wedge\boldsymbol{V})\,\mathrm{d}S=\int_v(\boldsymbol{\Omega}\wedge\boldsymbol{V})\wedge\boldsymbol{x}\,\mathrm{d}v+\int_v\boldsymbol{x}\wedge\boldsymbol{E}(\boldsymbol{\Omega}\wedge\boldsymbol{x})\,\mathrm{d}v. \tag{C 5}$$

Using then the relationship

$$\boldsymbol{\Omega}\wedge\int_v(\boldsymbol{x}\wedge\boldsymbol{V})\mathrm{d}v+\int_v(\boldsymbol{\Omega}\wedge\boldsymbol{V})\wedge\boldsymbol{x}\,\mathrm{d}v=\int_v(\boldsymbol{\Omega}\wedge\boldsymbol{x})\wedge\boldsymbol{V}\mathrm{d}v, \tag{C 6}$$

which follows from the Jacobi identity

$$\boldsymbol{\Omega}\wedge(\boldsymbol{x}\wedge\boldsymbol{V})+\boldsymbol{x}\wedge(\boldsymbol{V}\wedge\boldsymbol{\Omega})+\boldsymbol{V}\wedge(\boldsymbol{\Omega}\wedge\boldsymbol{x})=0, \tag{C 7}$$

we finally obtain

$$\frac{\mathrm{d}^*}{\mathrm{d}t}\int_S\phi(\boldsymbol{x}\wedge\boldsymbol{n})\,\mathrm{d}S=\int_v(\boldsymbol{\Omega}\wedge\boldsymbol{x})\wedge\boldsymbol{V}\mathrm{d}v+\int_v\boldsymbol{x}\wedge(\boldsymbol{E}\hat{\boldsymbol{\Omega}}\boldsymbol{x})\,\mathrm{d}v$$
$$+\int_S(\boldsymbol{x}\wedge\boldsymbol{n})\boldsymbol{V}\cdot\boldsymbol{U}\,\mathrm{d}S+\int_S(\boldsymbol{x}\wedge\boldsymbol{n})\frac{\partial\phi}{\partial t}\,\mathrm{d}S+\frac{\delta}{\delta t}\int_S(\boldsymbol{x}\wedge\boldsymbol{n})\phi\,\mathrm{d}S. \tag{C 8}$$

The first integral in the left-hand side of (C 2), i.e.

$$\frac{\mathrm{d}^*}{\mathrm{d}t}\int_S\acute{\phi}(\boldsymbol{x}\wedge\boldsymbol{n})\,\mathrm{d}S,$$

can be treated by the same approach as outlined in Appendix B, §B.1, by simply replacing $\boldsymbol{\Phi}$ by $\boldsymbol{\Psi}$. Such a procedure leads to

$$\boldsymbol{M}_{un} = -\boldsymbol{M}^{(q)} + \mathbf{C}_{un}\boldsymbol{U} + \mathbf{D}_{un}\boldsymbol{\Omega} + \int_v \boldsymbol{x} \wedge \frac{\partial \boldsymbol{V}}{\partial t}\, \mathrm{d}v - \int_S \boldsymbol{z}\frac{\partial \boldsymbol{V}}{\partial t}\, \mathrm{d}S + \frac{\delta}{\delta t}\left(\int_v \boldsymbol{x} \wedge \boldsymbol{V}\, \mathrm{d}v - \int_S \boldsymbol{z}\boldsymbol{V}\, \mathrm{d}S\right). \tag{C 9}$$

Using the above, one can also derive the following expressions for $\mathbf{C}_{un}\boldsymbol{U}$ and $\mathbf{D}_{un}\boldsymbol{\Omega}$:

$$\mathbf{C}_{un}\boldsymbol{U} = \int_S \boldsymbol{m}\boldsymbol{V}\, \mathrm{d}S \wedge \boldsymbol{U} - \int_S \boldsymbol{z}\mathbf{E}\boldsymbol{U}\, \mathrm{d}S + \int_S (\boldsymbol{x} \wedge \boldsymbol{n})(\boldsymbol{V} \cdot \boldsymbol{U})\, \mathrm{d}S + \boldsymbol{U} \wedge \int_v \boldsymbol{V} \mathrm{d}v, \tag{C 10}$$

and

$$\mathbf{D}_{un}\boldsymbol{\Omega} = \int_S (\boldsymbol{z}(\boldsymbol{\Omega} \wedge \boldsymbol{V}) - \boldsymbol{\Omega} \wedge \boldsymbol{z}\boldsymbol{V})\, \mathrm{d}S - \int_S \boldsymbol{z}\mathbf{E}(\boldsymbol{\Omega} \wedge \boldsymbol{x})\, \mathrm{d}S \tag{C 11}$$

$$+ \int_v ((\boldsymbol{\Omega} \wedge \boldsymbol{x}) \wedge \boldsymbol{V} + \boldsymbol{x} \wedge \mathbf{E}(\boldsymbol{\Omega} \wedge \boldsymbol{x}))\, \mathrm{d}v. \tag{C 12}$$

As demonstrated in what follows, the last two terms of (C 10) and the last two terms of (C 12) precisely cancel the corresponding terms in the expression for the steady part of the moment.

C.2. *The steady moment*

Employing for the steady part of the moment the same approach previously used for evaluating the steady part of the force yields

$$\boldsymbol{M}_{st} = \int_S \phi' \frac{\partial}{\partial n}(\boldsymbol{x} \wedge \boldsymbol{V})\, \mathrm{d}S - \int_S \frac{\partial \phi'}{\partial n}(\boldsymbol{x} \wedge \boldsymbol{V})\, \mathrm{d}S, \tag{C 13}$$

from where, by subtracting the $\boldsymbol{U}$-part from (2.19) one gets

$$\mathbf{C}_{st}\boldsymbol{U} = \int_S (\boldsymbol{U} \cdot \boldsymbol{\Phi})(\boldsymbol{n} \wedge \boldsymbol{V})\, \mathrm{d}S + \int_S (\boldsymbol{U} \cdot \boldsymbol{\Phi})(\boldsymbol{x} \wedge \mathbf{E}\boldsymbol{n})\, \mathrm{d}S - \int_S (\boldsymbol{U} \cdot \boldsymbol{n})(\boldsymbol{x} \wedge \boldsymbol{V})\, \mathrm{d}S. \tag{C 14}$$

Adding then (C 10), (C 12) and (C 14), and noting that $\mathbf{C} = \mathbf{C}_{un} + \mathbf{C}_{st}$, renders

$$\mathbf{C}\boldsymbol{U} = \int_S \left((\boldsymbol{m}\boldsymbol{V}) \wedge \boldsymbol{U} - \boldsymbol{V} \wedge (\boldsymbol{m}\boldsymbol{U}) + \boldsymbol{x} \wedge (\mathbf{E}\boldsymbol{m}^T\boldsymbol{U}) - \boldsymbol{z}\mathbf{E}\boldsymbol{U}\right) \mathrm{d}S, \tag{C 15}$$

since (C 10) cancels the last integral in the right-hand side of (C 14).

Similar considerations for the contributions of the $\boldsymbol{\Omega}$-terms in (2.19) lead to

$$\mathbf{D}_{st}\boldsymbol{\Omega} = \int_S (\boldsymbol{\Omega} \cdot \boldsymbol{\Psi})\frac{\partial}{\partial n}(\boldsymbol{x} \wedge \boldsymbol{V})\, \mathrm{d}S - \int_S (\boldsymbol{\Omega} \cdot (\boldsymbol{x} \wedge \boldsymbol{n}))(\boldsymbol{x} \wedge \boldsymbol{V})\, \mathrm{d}S. \tag{C 16}$$

Here again the last integral in (C 16) cancels the corresponding term in the expression for $\mathbf{D}_{un}\boldsymbol{\Omega}$ (C 12). Indeed, one gets

$$\int_S \boldsymbol{\Omega} \cdot (\boldsymbol{x} \wedge \boldsymbol{n})(\boldsymbol{x} \wedge \boldsymbol{V})\, \mathrm{d}S = \int_v ((\boldsymbol{\Omega} \wedge \boldsymbol{x}) \cdot \nabla)\boldsymbol{x} \wedge \boldsymbol{V}\, \mathrm{d}v = \int_v ((\boldsymbol{\Omega} \wedge \boldsymbol{x}) \wedge \boldsymbol{V} + \boldsymbol{x} \wedge \mathbf{E}(\boldsymbol{\Omega} \wedge \boldsymbol{x}))\, \mathrm{d}v, \tag{C 17}$$

which is identical with (C 12).

Finally, by adding (C 11), (C 12) and (C 16), we find that

$$\boldsymbol{D\Omega} = \left(\int_S \left(\boldsymbol{\Psi} \wedge \frac{\partial}{\partial n}(\boldsymbol{x} \wedge \boldsymbol{V}) + \boldsymbol{z}\boldsymbol{V} \right) \, \mathrm{d}S \right) \wedge \boldsymbol{\Omega}, \tag{C 18}$$

in full agreement with (3.20) and (3.21).

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